

Gaussian-Perturbative Calculations with a Homogeneous External Source

Jorge L. deLyra
Department of Mathematical Physics
Physics Institute
University of São Paulo

March 17, 2014

Abstract

We derive the equation of the critical curve and calculate the renormalized masses of the $SO(\mathfrak{N})$ -symmetric $\lambda\phi^4$ model in the presence of a homogeneous external source. We do this using the Gaussian-Perturbative approximation on finite lattices and explicitly taking the continuum limit. No disabling divergences are found in the final results, and no renormalization is necessary. We show that the results give a complete description of the critical behavior of the model and of the phenomenon of spontaneous symmetry breaking, at the quantum-field-theoretical level.

We show that the renormalized masses depend on the external source, and point out the consequences of that fact for the design of computer simulations of the model. We point out a simple but interesting consequence of the results, regarding the role of the $\lambda\phi^4$ model in the Standard Model of high-energy particle physics. Using the experimentally known values of the mass and of the expectation value of the Higgs field, we determine uniquely the values of the *bare* dimensionless parameters α and λ of the model, which turn out to be small numbers, significantly less than one.

1 Introduction

Years ago we introduced a calculational technique that was quite successful in describing the critical behavior of the $SO(\mathfrak{N})$ -symmetric Euclidean $\lambda\phi^4$ model in $d \geq 3$ spacetime dimensions [1]. Some quantities were calculated in $d = 4$ and compared to the results of computer simulations, yielding surprisingly good results, and describing reliably the most important qualitative aspects of the model. The observables calculated were the expectation value of the field, which is the order parameter of the critical transition of the model and describes the phenomenon of spontaneous symmetry breaking, and the two-point function, from which one can get the renormalized masses and hence the correlation lengths, in both phases of the model.

Although inspired by and superficially similar to perturbation theory, the technique can handle a phenomenon such as spontaneous symmetry breaking, which is usually considered to be out of reach for plain perturbation theory. The innovative and essential aspect of the technique is the use of certain self-consistency conditions within a framework similar to that of perturbation theory. The technique would be better described as a Gaussian approximation rather than a perturbative expansion. As such, it is able to produce good predictions for the one-point and two-point observables, since these are the moments present

in the Gaussian distribution, but should not be expected to go much further than that. For lack of a better name, we shall refer to it as the Gaussian-Perturbative approximation.

The important role that the four-component $\lambda\phi^4$ model plays in the Standard Model of high-energy particle physics makes it certainly interesting to learn more about it. In this paper we extend the Gaussian-Perturbative technique introduced in [1] to the same model in the presence of external sources. These external sources are not thought of merely as analytical devices used to extract the Green's functions from the functional generators of the model, and to be put to zero afterwards. Instead, they are thought of as actual physical sources of particles in the model. One important objective is to determine how the introduction of the external sources affects the values of the renormalized masses in either phase of the model.

These are analytical calculations performed on the Euclidean lattice, which therefore allow us to discuss, and to explicitly take, specific continuum limits in the quantum theory. As we will see, there is no need for perturbative renormalization, or for any regulation mechanism other than the lattice where the model is defined. All calculations on finite lattices are ordinary straightforward manipulations. Although there are some quantities that do diverge in the continuum limit, they all cancel off from the observables before the limit is taken.

2 The Model

Let us start by giving the definition of the model, in the classical and quantum domains, and then quickly reviewing the Gaussian-Perturbative approximation. Consider then the Euclidean quantum field theories of an $SO(\mathfrak{N})$ -symmetric set of scalar fields $\vec{\phi}(x_\mu)$ defined within a periodical cubic box of side L in d dimensions by the classical action

$$S[\vec{\phi}] = \oint_{L^d} d^d x \left\{ \frac{1}{2} \sum_\nu \left[\partial_\nu \vec{\phi}(x_\mu) \cdot \partial_\nu \vec{\phi}(x_\mu) \right] + \frac{m^2}{2} \left[\vec{\phi}(x_\mu) \cdot \vec{\phi}(x_\mu) \right] + \frac{\Lambda}{4} \left[\vec{\phi}(x_\mu) \cdot \vec{\phi}(x_\mu) \right]^2 - J_0 \phi_{\mathfrak{N}}(x_\mu) \right\},$$

where $d \geq 3$. This is the usual form of the $SO(\mathfrak{N})$ -symmetric $\lambda\phi^4$ model in the classical continuum, with an external source J_0 , which by assumption is a constant. The vector notation $\vec{\phi}(x_\mu)$ is shorthand for

$$\vec{\phi}(x_\mu) = (\phi_1(x_\mu), \phi_2(x_\mu), \dots, \phi_{\mathfrak{N}}(x_\mu)),$$

and the dot-product notation represents the scalar product of vectors in the internal $SO(\mathfrak{N})$ space, that is a sum over $i = 1, 2, \dots, \mathfrak{N}$,

$$\vec{\phi}(x_\mu) \cdot \vec{\phi}(x_\mu) = \sum_i^{\mathfrak{N}} [\phi_i(x_\mu)]^2.$$

In this action the quantity J_0 is a homogeneous external source associated with the $\phi_{\mathfrak{N}}(x_\mu)$ field component. Its introduction breaks the $SO(\mathfrak{N})$ symmetry, of course, and causes the generation of a non-zero expectation value for the $\phi_{\mathfrak{N}}(x_\mu)$ field component.

In order to use the definition of the quantum theory on a cubical lattice of size L with N sites along each direction, with lattice spacing $a = L/N$, we consider the corresponding lattice action

$$\begin{aligned}
S_N[\vec{\varphi}] = & \sum_{n_\mu}^{N^d} \left\{ \frac{1}{2} \sum_{\nu}^d [\Delta_\nu \vec{\varphi}(n_\mu) \cdot \Delta_\nu \vec{\varphi}(n_\mu)] + \frac{\alpha}{2} [\vec{\varphi}(n_\mu) \cdot \vec{\varphi}(n_\mu)] + \right. \\
& \left. + \frac{\lambda}{4} [\vec{\varphi}(n_\mu) \cdot \vec{\varphi}(n_\mu)]^2 - j_0 \varphi_{\mathfrak{N}}(n_\mu) \right\},
\end{aligned}$$

where all quantities are now dimensionless, defined by the appropriate scalings,

$$\begin{aligned}
\varphi_i(n_\mu) &= a^{(d-2)/2} \phi_i(x_\mu), \\
n_\mu &= a^{-1} x_\mu, \\
\alpha &= a^2 m^2, \\
\lambda &= a^{4-d} \Lambda, \\
j_0 &= a^{(d+2)/2} J_0.
\end{aligned} \tag{1}$$

In order for the model to be stable we must have $\lambda \geq 0$ and, in addition to this, if $\lambda = 0$ then we must also have $\alpha \geq 0$. Up to this point there are no further constraints on the real parameters α and λ .

When possible, the summations are notated, in the subscript, by the variable which is being summed over, and, in the superscript, by the number of terms in the sum. The integer coordinates n_μ are taken to vary as symmetrically as possible around the origin $n_\mu = 0_\mu$, that is we have $n_\mu = n_{\min}, \dots, 0, \dots, n_{\max}$ with certain values of n_{\min} and n_{\max} that depend on the parity of N ,

$$n_\mu = -\frac{N-1}{2}, \dots, 0, \dots, \frac{N-1}{2},$$

for odd N , and

$$n_\mu = -\frac{N}{2} + 1, \dots, 0, \dots, \frac{N}{2},$$

for even N , in either case for all values of $\mu = 1, \dots, d$.

In this paper we will perform the calculations of the critical line and of the renormalized masses in a situation in which we have, in terms of the dimensionfull field $\vec{\phi}(x_\mu)$, for $i = 1, \dots, \mathfrak{N} - 1$,

$$\langle \phi_i(x_\mu) \rangle = 0,$$

and, for $i = \mathfrak{N}$,

$$\langle \phi_{\mathfrak{N}}(x_\mu) \rangle = V_0,$$

where V_0 is a constant with the physical dimensions of the field $\phi_{\mathfrak{N}}(x_\mu)$. In terms of the dimensionless field $\vec{\varphi}(n_\mu)$ we have for the only non-trivial condition

$$\langle \varphi_{\mathfrak{N}}(n_\mu) \rangle = v_0,$$

where the dimensionless constant is given by $v_0 = a^{(d-2)/2} V_0$.

We will consider continuum limits in which we have both $N \rightarrow \infty$ and $L \rightarrow \infty$. In order to do this we will choose to make L increase as \sqrt{N} and a decrease as \sqrt{N} , so that we still have $a = L/N$. The calculations on finite lattices will be performed with periodical

boundary conditions, with the understanding that at the end of the day such a limit is to be taken.

Observe that we are specifying the value of the expectation value v_0 of $\varphi_{\mathfrak{N}}(n_\mu)$ rather than the value of the corresponding external source j_0 . What we are doing here is to assume that there is some external source present such that we have the expectation value specified. It follows that one of the expected results of our calculations is the determination, at least implicitly, of the form of the external source in terms of v_0 .

Our first calculational task in preparation for the Gaussian-Perturbative calculations is to rewrite the action in terms of a shifted field, which has a null expectation value. We thus define a new field variable $\vec{\varphi}'(n_\mu)$ such that

$$\vec{\varphi}(n_\mu) = \vec{\varphi}'(n_\mu) + (0, 0, \dots, v_0),$$

so that we have $\langle \vec{\varphi}'(n_\mu) \rangle = 0$ for all n_μ , with $\mu = 1, \dots, d$. We must now determine the form of the action in terms of $\vec{\varphi}'(n_\mu)$. If we write each term of the action in terms of the shifted field we get

$$\begin{aligned} S_N[\vec{\varphi}'] = & \sum_{n_\mu}^{N^d} \left\{ \frac{1}{2} \sum_{\nu}^d [\Delta_{\nu} \vec{\varphi}'(n_\mu) \cdot \Delta_{\nu} \vec{\varphi}'(n_\mu)] + \right. \\ & + \frac{\alpha}{2} [\vec{\varphi}'(n_\mu) \cdot \vec{\varphi}'(n_\mu)] + \alpha v_0 \varphi'_{\mathfrak{N}}(n_\mu) + \frac{\alpha}{2} v_0^2 + \\ & + \frac{\lambda}{4} [\vec{\varphi}'(n_\mu) \cdot \vec{\varphi}'(n_\mu)]^2 + \lambda v_0 [\vec{\varphi}'(n_\mu) \cdot \vec{\varphi}'(n_\mu)] \varphi'_{\mathfrak{N}}(n_\mu) + \\ & + \frac{\lambda}{2} v_0^2 [\vec{\varphi}'(n_\mu) \cdot \vec{\varphi}'(n_\mu)] + \lambda v_0^2 \varphi'^2_{\mathfrak{N}}(n_\mu) + \\ & + \lambda v_0^3 \varphi'_{\mathfrak{N}}(n_\mu) + \frac{\lambda}{4} v_0^4 + \\ & \left. - j_0 \varphi'_{\mathfrak{N}}(n_\mu) - j_0 v_0 \right\}. \end{aligned}$$

We will now eliminate all field-independent terms, since they correspond to constant factors that cancel off in the ratios of functional integrals which give the expectation values of the observables. Doing this we get the equivalent action

$$\begin{aligned} S_N[\vec{\varphi}'] = & \sum_{n_\mu}^{N^d} \left\{ \frac{1}{2} \sum_{\nu}^d [\Delta_{\nu} \vec{\varphi}'(n_\mu) \cdot \Delta_{\nu} \vec{\varphi}'(n_\mu)] + \right. \\ & + \alpha v_0 \varphi'_{\mathfrak{N}}(n_\mu) + \lambda v_0^3 \varphi'_{\mathfrak{N}}(n_\mu) - j_0 \varphi'_{\mathfrak{N}}(n_\mu) + \\ & + \frac{\alpha + \lambda v_0^2}{2} [\vec{\varphi}'(n_\mu) \cdot \vec{\varphi}'(n_\mu)] + \lambda v_0^2 \varphi'^2_{\mathfrak{N}}(n_\mu) + \\ & \left. + \lambda v_0 [\vec{\varphi}'(n_\mu) \cdot \vec{\varphi}'(n_\mu)] \varphi'_{\mathfrak{N}}(n_\mu) + \frac{\lambda}{4} [\vec{\varphi}'(n_\mu) \cdot \vec{\varphi}'(n_\mu)]^2 \right\}, \end{aligned}$$

where except for the kinetic part the terms have been ordered by increasing powers of the field.

The last task we have to perform, in preparation for the Gaussian-Perturbative calculations, is the separation of the action in two parts. Since the symmetry is broken by

the introduction of the external sources, besides the fact that depending on the values of the parameters α and λ it might be spontaneously broken as well, this separation involves two new mass parameters, α_0 for $\varphi'_1(n_\mu), \dots, \varphi'_{\mathfrak{N}-1}(n_\mu)$, and $\alpha_{\mathfrak{N}}$ for $\varphi'_{\mathfrak{N}}(n_\mu)$. Note that an $SO(\mathfrak{N}-1)$ symmetry subgroup is left over after the $SO(\mathfrak{N})$ symmetry breakdown. We therefore adopt for the Gaussian part of the action

$$S_0[\vec{\varphi}'] = \sum_{n_\mu}^{N^d} \left\{ \frac{1}{2} \sum_{\nu}^d [\Delta_{\nu} \vec{\varphi}'(n_\mu) \cdot \Delta_{\nu} \vec{\varphi}'(n_\mu)] + \frac{\alpha_0}{2} [\vec{\varphi}'(n_\mu) \cdot \vec{\varphi}'(n_\mu)] + \frac{\alpha_{\mathfrak{N}} - \alpha_0}{2} \varphi'^2_{\mathfrak{N}}(n_\mu) \right\}, \quad (2)$$

where there are no constraints on the parameters introduced other than $\alpha_0 \geq 0$ and $\alpha_{\mathfrak{N}} \geq 0$. Note that, despite the way in which this is written, we do in fact have here just an α_0 mass term for each field component $\varphi'_i(n_\mu)$, for $i = 1, \dots, \mathfrak{N}-1$, and an $\alpha_{\mathfrak{N}}$ mass term for the field component $\varphi'_{\mathfrak{N}}(n_\mu)$. It follows that the non-Gaussian part of the action is

$$S_V[\vec{\varphi}'] = \sum_{n_\mu}^{N^d} \left\{ v_0 [\alpha + \lambda v_0^2] \varphi'_{\mathfrak{N}}(n_\mu) - j_0 \varphi'_{\mathfrak{N}}(n_\mu) + \frac{\alpha - \alpha_0 + \lambda v_0^2}{2} [\vec{\varphi}'(n_\mu) \cdot \vec{\varphi}'(n_\mu)] + \frac{\alpha_0 - \alpha_{\mathfrak{N}} + 2\lambda v_0^2}{2} \varphi'^2_{\mathfrak{N}}(n_\mu) + \lambda v_0 [\vec{\varphi}'(n_\mu) \cdot \vec{\varphi}'(n_\mu)] \varphi'_{\mathfrak{N}}(n_\mu) + \frac{\lambda}{4} [\vec{\varphi}'(n_\mu) \cdot \vec{\varphi}'(n_\mu)]^2 \right\}, \quad (3)$$

which has its terms now written strictly in the order of increasing powers of the field.

Let us end this section by recalling the calculational techniques that will be involved. Given an arbitrary observable $\mathbf{O}[\vec{\varphi}']$ its expectation value is defined by

$$\langle \mathbf{O}[\vec{\varphi}'] \rangle = \frac{\int [d\varphi] \mathbf{O}[\vec{\varphi}'] e^{-S_0[\vec{\varphi}'] - \xi S_V[\vec{\varphi}']}}{\int [d\varphi] e^{-S_0[\vec{\varphi}'] - \xi S_V[\vec{\varphi}']}},$$

which is a function of ξ , where $[d\varphi]$ denotes the flat measure and hence integrals from $-\infty$ to $+\infty$ over all the field components at all sites. The expectation values of the model are obtained for $\xi = 1$, and the corresponding expectation values in the Gaussian measure of $S_0[\vec{\varphi}']$ are those obtained for $\xi = 0$. The Gaussian-Perturbative approximation consists of the expansion of the right-hand side in powers of ξ to some finite order, around the point $\xi = 0$, and the application of the resulting expression at $\xi = 1$. The first-order Gaussian-Perturbative approximation of the expectation value of the observable $\mathbf{O}[\vec{\varphi}']$ is given by

$$\langle \mathbf{O}[\vec{\varphi}'] \rangle = \langle \mathbf{O}[\vec{\varphi}'] \rangle_0 - \{ \langle \mathbf{O}[\vec{\varphi}'] S_V[\vec{\varphi}'] \rangle_0 - \langle \mathbf{O}[\vec{\varphi}'] \rangle_0 \langle S_V[\vec{\varphi}'] \rangle_0 \},$$

where the subscript 0 indicates the expectation values in the measure of $S_0[\vec{\varphi}']$. These expectation values are most easily calculated in momentum space, where they involve only uncoupled Gaussian integrals. Therefore, let us also recall here the transformations to and from the momentum space representation of the model. We have for the field and its Fourier transform $\tilde{\varphi}'_i(k_\mu)$

$$\begin{aligned}\tilde{\varphi}'_i(k_\mu) &= \frac{1}{N^d} \sum_{n_\mu}^{N^d} e^{\mathbf{i}(2\pi/N) \sum_\mu k_\mu n_\mu} \varphi'_i(n_\mu), \\ \varphi'_i(n_\mu) &= \sum_{k_\mu}^{N^d} e^{-\mathbf{i}(2\pi/N) \sum_\mu k_\mu n_\mu} \tilde{\varphi}'_i(k_\mu),\end{aligned}$$

where the sums over k_μ are taken in as symmetric a way as possible around $k_\mu = 0_\mu$, just as we did for n_μ . In other words, we have $k_\mu = k_{\min}, \dots, 0, \dots, k_{\max}$ with the same values of k_{\min} and k_{\max} , depending on the parity of N , that were used for n_{\min} and n_{\max} ,

$$k_\mu = -\frac{N-1}{2}, \dots, 0, \dots, \frac{N-1}{2},$$

for odd N , and

$$k_\mu = -\frac{N}{2} + 1, \dots, 0, \dots, \frac{N}{2},$$

for even N , in either case for all values of $\mu = 1, \dots, d$. The orthogonality and completeness relations of the Fourier base are given by

$$\begin{aligned}\sum_{n_\mu}^{N^d} e^{\pm \mathbf{i}(2\pi/N) \sum_\mu n_\mu (k_\mu - k'_\mu)} &= N^d \delta^d(k_\mu, k'_\mu), \\ \sum_{k_\mu}^{N^d} e^{\pm \mathbf{i}(2\pi/N) \sum_\mu k_\mu (n_\mu - n'_\mu)} &= N^d \delta^d(n_\mu, n'_\mu).\end{aligned}$$

A typical Gaussian expectation value in momentum space, and possibly the most fundamental one, is given for a generic field component by

$$\langle \tilde{\varphi}'_i(k_\mu) \tilde{\varphi}'_i{}^*(k_\mu) \rangle_0 = \frac{1}{N^d} \frac{1}{\rho^2(k_\mu) + \alpha_i},$$

where α_i is either α_0 or $\alpha_{\mathfrak{N}}$, depending on the field component involved, and where $\rho^2(k_\mu)$ are the eigenvalues of the discrete Laplacian on the lattice, which are given by

$$\rho^2(k_\mu) = 4 \left[\sin^2\left(\frac{\pi k_1}{N}\right) + \dots + \sin^2\left(\frac{\pi k_d}{N}\right) \right].$$

This and several other expectation values, Gaussian integration formulas and lattice sums can be found in Appendix B.

3 Calculations

We are now ready for the Gaussian-Perturbative calculations. We start with the calculations involving the critical behavior of the model, so that we may determine and characterize its two phases. It is important to observe that the phase structure of the model must be established right at the beginning, because everything else has to be discussed in terms of it.

3.1 The Critical Line

We will now calculate the Gaussian-Perturbative approximation for the particular observable $\mathbf{O}[\vec{\varphi}'] = \varphi_{\mathfrak{N}}(n'_\mu)$, at some arbitrary point n'_μ . If we write the observable in terms of the shifted field we get

$$\begin{aligned}\mathbf{O}[\vec{\varphi}'] &= \varphi_{\mathfrak{N}}(n'_\mu) \\ &= \varphi'_{\mathfrak{N}}(n'_\mu) + v_0.\end{aligned}$$

In order to get the equation of the critical line we impose, in a self-consistent way, that we in fact have

$$\langle \varphi_{\mathfrak{N}}(n'_\mu) \rangle = v_0,$$

which is the same as stating that

$$\langle \varphi'_{\mathfrak{N}}(n'_\mu) \rangle = 0.$$

In the first-order Gaussian-Perturbative approximation this becomes

$$\begin{aligned}\langle \varphi'_{\mathfrak{N}}(n'_\mu) \rangle &= \langle \varphi'_{\mathfrak{N}}(n'_\mu) \rangle_0 - \left\{ \langle \varphi'_{\mathfrak{N}}(n'_\mu) S_V[\vec{\varphi}'] \rangle_0 - \langle \varphi'_{\mathfrak{N}}(n'_\mu) \rangle_0 \langle S_V[\vec{\varphi}'] \rangle_0 \right\} \\ &= 0.\end{aligned}$$

Since we have $\langle \varphi'_{\mathfrak{N}}(n'_\mu) \rangle_0 = 0$, because this observable is field-odd and the Gaussian action $S_0[\vec{\varphi}']$ is field-even, we get for the critical line the simple equation, known as the tadpole equation,

$$\langle \varphi'_{\mathfrak{N}}(n'_\mu) S_V[\vec{\varphi}'] \rangle_0 = 0.$$

The expectation value shown here is calculated in Appendix A, given in Equation (A.4), and the result is

$$\langle \varphi'_{\mathfrak{N}}(n'_\mu) S_V[\vec{\varphi}'] \rangle_0 = \frac{v_0 [\alpha + v_0^2 \lambda + (\mathfrak{N} - 1) \lambda \sigma_0^2 + 3 \lambda \sigma_{\mathfrak{N}}^2] - j_0}{\alpha_{\mathfrak{N}}}.$$

The parameter $\alpha_{\mathfrak{N}}$ cancels off from our equation, and thus we are left with the result

$$j_0 = v_0 \left\{ \lambda v_0^2 + \alpha + \lambda [(\mathfrak{N} - 1) \sigma_0^2 + 3 \sigma_{\mathfrak{N}}^2] \right\}, \quad (4)$$

in which we now isolated on the left-hand side the term with the external source. This gives the general relation between j_0 and v_0 at each point (α, λ) of the parameter space of the model. As we shall see later, from this result we can determine the critical behavior of the model and derive the equation of the critical line.

The quantity σ_0 is the width or variance of the local distribution of values of the field components $\varphi'_i(n_\mu)$, with $i = 1, \dots, \mathfrak{N} - 1$, in the measure of $S_0[\vec{\varphi}']$,

$$\begin{aligned}\sigma_0^2 &= \langle \varphi_i'^2(n_\mu) \rangle_0, \\ &= \frac{1}{N^d} \sum_{k_\mu} \frac{1}{\rho^2(k_\mu) + \alpha_0},\end{aligned}$$

as one can see in Appendix B, Equation (B.6), and has the following interesting properties, so long as $d \geq 3$. First, it is independent of the position n_μ , as translation invariance would require. Second, for $d \geq 3$ it has a finite and non-zero $N \rightarrow \infty$ limit, so long as $\alpha_0 = a^2 m_0^2$ with a finite value of m_0 in the limit. Finally, the value of σ_0 in the limit does not depend on the value of m_0 in that same limit. Analogously, the quantity $\sigma_{\mathfrak{N}}$ is associated to the remaining field component $\varphi'_{\mathfrak{N}}(n_\mu)$ and to the mass parameter $\alpha_{\mathfrak{N}}$, and has these same properties. In fact, σ_0 and $\sigma_{\mathfrak{N}}$ have exactly the same value in the $N \rightarrow \infty$ limit.

3.2 The Transversal Propagator

We will now calculate the expectation value of the observable

$$\mathbf{O}[\vec{\varphi}] = \varphi'_i(n'_\mu) \varphi'_i(n''_\mu),$$

which has the same form for all components of the field except $i = \mathfrak{N}$. We call this the transversal propagator because it belongs to the field components which are orthogonal to the direction of the external source in the internal $SO(\mathfrak{N})$ space. In this section we will assume that $i \neq \mathfrak{N}$, in fact we will make $i = 1$. The observable will be taken at two arbitrary points n'_μ and n''_μ . The first-order Gaussian-Perturbative approximation for this observable gives

$$\begin{aligned} \langle \varphi'_1(n'_\mu) \varphi'_1(n''_\mu) \rangle &= \langle \varphi'_1(n'_\mu) \varphi'_1(n''_\mu) \rangle_0 + \\ &\quad - \left\{ \langle \varphi'_1(n'_\mu) \varphi'_1(n''_\mu) S_V[\vec{\varphi}] \rangle_0 - \langle \varphi'_1(n'_\mu) \varphi'_1(n''_\mu) \rangle_0 \langle S_V[\vec{\varphi}] \rangle_0 \right\} \\ &= g_0(n'_\mu - n''_\mu) + \\ &\quad - \left\{ \langle \varphi'_1(n'_\mu) \varphi'_1(n''_\mu) S_V[\vec{\varphi}] \rangle_0 - g_0(n'_\mu - n''_\mu) \langle S_V[\vec{\varphi}] \rangle_0 \right\}, \end{aligned}$$

where $g_0(n'_\mu - n''_\mu)$ is the two-point function with mass parameter α_0 . We must calculate the two expectation values which appear in this formula. The calculation of the first one is done in Appendix A, given in Equation (A.5), and results in

$$\begin{aligned} \langle S_V[\vec{\varphi}] \rangle_0 &= N^d \left[\frac{\alpha - \alpha_0 + \lambda v_0^2}{2} (\mathfrak{N} - 1) \sigma_0^2 + \frac{\alpha - \alpha_{\mathfrak{N}} + 3\lambda v_0^2}{2} \sigma_{\mathfrak{N}}^2 + \right. \\ &\quad \left. + \frac{\lambda}{4} (\mathfrak{N}^2 - 1) \sigma_0^4 + \frac{\lambda}{2} (\mathfrak{N} - 1) \sigma_0^2 \sigma_{\mathfrak{N}}^2 + \frac{3\lambda}{4} \sigma_{\mathfrak{N}}^4 \right]. \end{aligned}$$

The second expectation value is also calculated in Appendix A, given in Equation (A.6), and the result is

$$\begin{aligned} &\langle \varphi'_1(n'_\mu) \varphi'_1(n''_\mu) S_V[\vec{\varphi}] \rangle_0 \\ &= N^d \left[\frac{\alpha - \alpha_0 + \lambda v_0^2}{2} (\mathfrak{N} - 1) \sigma_0^2 + \frac{\alpha - \alpha_{\mathfrak{N}} + 3\lambda v_0^2}{2} \sigma_{\mathfrak{N}}^2 + \right. \\ &\quad \left. + \frac{\lambda}{4} (\mathfrak{N}^2 - 1) \sigma_0^4 + \frac{\lambda}{2} (\mathfrak{N} - 1) \sigma_0^2 \sigma_{\mathfrak{N}}^2 + \frac{3\lambda}{4} \sigma_{\mathfrak{N}}^4 \right] g_0(n'_\mu - n''_\mu) \\ &\quad + [\alpha - \alpha_0 + \lambda v_0^2 + \lambda(\mathfrak{N} + 1) \sigma_0^2 + \lambda \sigma_{\mathfrak{N}}^2] \frac{1}{N^d} \sum_{k_\mu}^{N^d} \frac{e^{-\mathbf{2}(2\pi/N) \sum_\mu k_\mu (n'_\mu - n''_\mu)}}{[\rho^2(k_\mu) + \alpha_0]^2}. \end{aligned}$$

The factor in front of $g_0(n'_\mu - n''_\mu)$ can now be verified to be exactly equal to $\langle S_V[\vec{\varphi}] \rangle_0$, and therefore this whole part cancels off from our observable. We may now write for the difference of expectation values that appears in it,

$$\begin{aligned} &\langle \varphi'_1(n'_\mu) \varphi'_1(n''_\mu) S_V[\vec{\varphi}] \rangle_0 - \langle S_V[\vec{\varphi}] \rangle_0 g_0(n'_\mu - n''_\mu) \\ &= [\lambda v_0^2 + \alpha - \alpha_0 + \lambda(\mathfrak{N} + 1) \sigma_0^2 + \lambda \sigma_{\mathfrak{N}}^2] \frac{1}{N^d} \sum_{k_\mu}^{N^d} \frac{e^{-\mathbf{2}(2\pi/N) \sum_\mu k_\mu (n'_\mu - n''_\mu)}}{[\rho^2(k_\mu) + \alpha_0]^2}. \end{aligned}$$

Finally, we can write the complete result,

$$\begin{aligned}
& \langle \varphi'_1(n'_\mu) \varphi'_1(n''_\mu) \rangle \\
&= \frac{1}{N^d} \sum_{k_\mu} \frac{e^{-\mathfrak{z}(2\pi/N) \sum_\mu^d k_\mu (n'_\mu - n''_\mu)}}{\rho^2(k_\mu) + \alpha_0} + \\
&\quad - [\lambda v_0^2 + \alpha - \alpha_0 + \lambda(\mathfrak{N} + 1)\sigma_0^2 + \lambda\sigma_{\mathfrak{N}}^2] \frac{1}{N^d} \sum_{k_\mu} \frac{e^{-\mathfrak{z}(2\pi/N) \sum_\mu^d k_\mu (n'_\mu - n''_\mu)}}{[\rho^2(k_\mu) + \alpha_0]^2} \\
&= \frac{1}{N^d} \sum_{k_\mu} \frac{e^{-\mathfrak{z}(2\pi/N) \sum_\mu^d k_\mu (n'_\mu - n''_\mu)}}{\rho^2(k_\mu) + \alpha_0} \left[1 - \frac{\lambda v_0^2 + \alpha - \alpha_0 + \lambda(\mathfrak{N} + 1)\sigma_0^2 + \lambda\sigma_{\mathfrak{N}}^2}{\rho^2(k_\mu) + \alpha_0} \right],
\end{aligned}$$

where we wrote $g_0(n'_\mu - n''_\mu)$ in terms of its Fourier transform.

In principle we could have used any positive value of α_0 for this calculation, but now a particular choice comes to our attention. We see from the structure of this propagator that we can make α_0 equal to the transversal renormalized mass parameter by choosing it so that the numerator of the second fraction vanishes. In this way we get a very simple propagator, with a simple pole in the complex ρ^2 plane, in which the parameter α_0 appears now in the role of the renormalized mass parameter,

$$\langle \varphi'_1(n'_\mu) \varphi'_1(n''_\mu) \rangle = \frac{1}{N^d} \sum_{k_\mu} \frac{e^{-\mathfrak{z}(2\pi/N) \sum_\mu^d k_\mu (n'_\mu - n''_\mu)}}{\rho^2(k_\mu) + \alpha_0}.$$

Observe that to this order the propagator is, in fact, the propagator of the free theory. This is a self-consistent way to choose the parameter α_0 , and is equivalent to the determination of the transversal renormalized mass. This choice is equivalent to requiring that the mass parameter of the Gaussian measure being used for the approximation of the expectation values be the same as the renormalized mass parameter of the original quantum model. It gives the result

$$\alpha_0 = \lambda v_0^2 + \alpha + \lambda [(\mathfrak{N} + 1)\sigma_0^2 + \sigma_{\mathfrak{N}}^2]. \quad (5)$$

This result for $\alpha_0 = a^2 m_0^2$ is valid for a constant but possibly non-zero external source, in both phases of the model, where m_0 is the mass associated to the $\mathfrak{N} - 1$ field components $\varphi'_i(n_\mu)$, for $i \neq \mathfrak{N}$.

3.3 The Longitudinal Propagator

We will now complete our calculations with the expectation value of the observable

$$\mathbf{O}[\vec{\varphi}] = \varphi'_{\mathfrak{N}}(n'_\mu) \varphi'_{\mathfrak{N}}(n''_\mu).$$

We call this the longitudinal propagator because it belongs to the field component which is in the direction of the external source in the internal $SO(\mathfrak{N})$ space. Once more the observable will be taken at two arbitrary points n'_μ and n''_μ . The first-order Gaussian-Perturbative approximation for this observable gives

$$\begin{aligned}
\langle \varphi'_{\mathfrak{N}}(n'_\mu) \varphi'_{\mathfrak{N}}(n''_\mu) \rangle &= \langle \varphi'_{\mathfrak{N}}(n'_\mu) \varphi'_{\mathfrak{N}}(n''_\mu) \rangle_0 + \\
&\quad - \left\{ \langle \varphi'_{\mathfrak{N}}(n'_\mu) \varphi'_{\mathfrak{N}}(n''_\mu) S_V[\vec{\varphi}] \rangle_0 - \langle \varphi'_{\mathfrak{N}}(n'_\mu) \varphi'_{\mathfrak{N}}(n''_\mu) \rangle_0 \langle S_V[\vec{\varphi}] \rangle_0 \right\} \\
&= g_{\mathfrak{N}}(n'_\mu - n''_\mu) + \\
&\quad - \left\{ \langle \varphi'_{\mathfrak{N}}(n'_\mu) \varphi'_{\mathfrak{N}}(n''_\mu) S_V[\vec{\varphi}] \rangle_0 - g_{\mathfrak{N}}(n'_\mu - n''_\mu) \langle S_V[\vec{\varphi}] \rangle_0 \right\},
\end{aligned}$$

where $g_{\mathfrak{N}}(n'_\mu - n''_\mu)$ is the two-point function with mass parameter $\alpha_{\mathfrak{N}}$. We must now calculate the two expectation values which appear in this formula. The first expectation value is the same we had before for the transversal propagator, and from Equation (A.5) we have,

$$\begin{aligned} \langle S_V[\vec{\varphi}] \rangle_0 &= N^d \left[\frac{\alpha - \alpha_0 + \lambda v_0^2}{2} (\mathfrak{N} - 1) \sigma_0^2 + \frac{\alpha - \alpha_{\mathfrak{N}} + 3\lambda v_0^2}{2} \sigma_{\mathfrak{N}}^2 + \right. \\ &\quad \left. + \frac{\lambda}{4} (\mathfrak{N}^2 - 1) \sigma_0^4 + \frac{\lambda}{2} (\mathfrak{N} - 1) \sigma_0^2 \sigma_{\mathfrak{N}}^2 + \frac{3\lambda}{4} \sigma_{\mathfrak{N}}^4 \right]. \end{aligned}$$

The second expectation value is calculated in Appendix A, given in Equation (A.7), and the result is

$$\begin{aligned} &\langle \varphi'_{\mathfrak{N}}(n'_\mu) \varphi'_{\mathfrak{N}}(n''_\mu) S_V[\vec{\varphi}] \rangle_0 \\ &= N^d \left[\frac{\alpha - \alpha_0 + \lambda v_0^2}{2} (\mathfrak{N} - 1) \sigma_0^2 + \frac{\alpha - \alpha_{\mathfrak{N}} + 3\lambda v_0^2}{2} \sigma_{\mathfrak{N}}^2 + \right. \\ &\quad \left. + \frac{\lambda}{4} (\mathfrak{N}^2 - 1) \sigma_0^4 + \frac{\lambda}{2} (\mathfrak{N} - 1) \sigma_0^2 \sigma_{\mathfrak{N}}^2 + \frac{3\lambda}{4} \sigma_{\mathfrak{N}}^4 \right] g_{\mathfrak{N}}(n'_\mu - n''_\mu) \\ &\quad + [\alpha - \alpha_{\mathfrak{N}} + 3\lambda v_0^2 + \lambda(\mathfrak{N} - 1) \sigma_0^2 + 3\lambda \sigma_{\mathfrak{N}}^2] \frac{1}{N^d} \sum_{k_\mu} \frac{e^{-\mathbf{i}(2\pi/N) \sum_\mu k_\mu (n'_\mu - n''_\mu)}}{[\rho^2(k_\mu) + \alpha_{\mathfrak{N}}]^2}. \end{aligned}$$

The factor in front of $g_{\mathfrak{N}}(n'_\mu - n''_\mu)$ can now be verified to be exactly equal to $\langle S_V[\vec{\varphi}] \rangle_0$, and therefore once again this whole part cancels off from our observable. We may now write for the difference of expectation values that appears in it,

$$\begin{aligned} &\langle \varphi'_{\mathfrak{N}}(n'_\mu) \varphi'_{\mathfrak{N}}(n''_\mu) S_V[\vec{\varphi}] \rangle_0 - \langle S_V[\vec{\varphi}] \rangle_0 g_{\mathfrak{N}}(n'_\mu - n''_\mu) \\ &= [3\lambda v_0^2 + \alpha - \alpha_{\mathfrak{N}} + \lambda(\mathfrak{N} - 1) \sigma_0^2 + 3\lambda \sigma_{\mathfrak{N}}^2] \frac{1}{N^d} \sum_{k_\mu} \frac{e^{-\mathbf{i}(2\pi/N) \sum_\mu k_\mu (n'_\mu - n''_\mu)}}{[\rho^2(k_\mu) + \alpha_{\mathfrak{N}}]^2}. \end{aligned}$$

Finally, we can write the complete result,

$$\begin{aligned} &\langle \varphi'_{\mathfrak{N}}(n'_\mu) \varphi'_{\mathfrak{N}}(n''_\mu) \rangle \\ &= \frac{1}{N^d} \sum_{k_\mu} \frac{e^{-\mathbf{i}(2\pi/N) \sum_\mu k_\mu (n'_\mu - n''_\mu)}}{\rho^2(k_\mu) + \alpha_{\mathfrak{N}}} + \\ &\quad - [3\lambda v_0^2 + \alpha - \alpha_{\mathfrak{N}} + \lambda(\mathfrak{N} - 1) \sigma_0^2 + 3\lambda \sigma_{\mathfrak{N}}^2] \frac{1}{N^d} \sum_{k_\mu} \frac{e^{-\mathbf{i}(2\pi/N) \sum_\mu k_\mu (n'_\mu - n''_\mu)}}{[\rho^2(k_\mu) + \alpha_{\mathfrak{N}}]^2} \\ &= \frac{1}{N^d} \sum_{k_\mu} \frac{e^{-\mathbf{i}(2\pi/N) \sum_\mu k_\mu (n'_\mu - n''_\mu)}}{\rho^2(k_\mu) + \alpha_{\mathfrak{N}}} \left[1 - \frac{3\lambda v_0^2 + \alpha - \alpha_{\mathfrak{N}} + \lambda(\mathfrak{N} - 1) \sigma_0^2 + 3\lambda \sigma_{\mathfrak{N}}^2}{\rho^2(k_\mu) + \alpha_{\mathfrak{N}}} \right], \end{aligned}$$

where we once more wrote $g_{\mathfrak{N}}(n'_\mu - n''_\mu)$ in terms of its Fourier transform. Exactly as in the previous case, we see from the structure of this propagator that we can make $\alpha_{\mathfrak{N}}$ equal to the longitudinal renormalized mass parameter by choosing it so that the numerator of the second fraction vanishes. This gives the result

$$\alpha_{\mathfrak{N}} = 3\lambda v_0^2 + \alpha + \lambda [(\mathfrak{N} - 1) \sigma_0^2 + 3\sigma_{\mathfrak{N}}^2]. \quad (6)$$

This result for $\alpha_{\mathfrak{N}} = a^2 m_{\mathfrak{N}}^2$ is valid for a constant but possibly non-zero external source, in both phases of the model, where $m_{\mathfrak{N}}$ is the mass associated to the field component $\varphi'_{\mathfrak{N}}(n_\mu)$.

4 Discussion

In this section we analyze and discuss the physical significance of the results obtained with the Gaussian-Perturbative approximation, starting with the determination of the critical behavior of the model. As was pointed out before, it is important that the phase structure of the model be established right at the beginning, because everything else has to be discussed in terms of it.

4.1 Critical Behavior

Critical Line: Here we discuss the physical significance of our result for j_0 as a function of v_0 . As we shall see, this bears on the critical behavior of the model. First of all, let us discuss the case $j_0 = 0$, that is without external sources at all, which from Equation (4) results in the equation

$$v_0 \{ \lambda v_0^2 + \alpha + \lambda [(\mathfrak{N} - 1)\sigma_0^2 + 3\sigma_{\mathfrak{N}}^2] \} = 0.$$

Observe that we do *not* assume that v_0 is automatically zero. Since we must have $\lambda \geq 0$, in the part of the parameter space of the model in which the quantity shown below is positive,

$$\alpha + \lambda [(\mathfrak{N} - 1)\sigma_0^2 + 3\sigma_{\mathfrak{N}}^2] > 0,$$

the only possible solution to the equation is in fact $v_0 = 0$. This is the *symmetric phase* of the model. On the other hand, in the complementary region of the parameter space of the model, in which that same quantity is negative,

$$\alpha + \lambda [(\mathfrak{N} - 1)\sigma_0^2 + 3\sigma_{\mathfrak{N}}^2] < 0,$$

and once again because we must have $\lambda \geq 0$, there are two other solutions besides the $v_0 = 0$ solution, given by

$$\lambda v_0^2 = - \{ \alpha + \lambda [(\mathfrak{N} - 1)\sigma_0^2 + 3\sigma_{\mathfrak{N}}^2] \}. \quad (7)$$

Let us observe that since $\lambda > 0$ we must have $\alpha < 0$ here. This is the *broken-symmetric phase* of the model, where these solutions corresponds to the local minima of the potential, while $v_0 = 0$ corresponds to the local maximum. If we look for the locus in the (α, λ) parameter plane of the model in which the $v_0 = 0$ solution becomes the *only* possible solution, we arrive at the equation

$$\alpha + \lambda [(\mathfrak{N} - 1)\sigma_0^2 + 3\sigma_{\mathfrak{N}}^2] = 0.$$

This is an equation giving λ in terms of α , which thus determines a certain curve in the parameter plane of the model, in this case a straight line. This is the *critical line*, which separates the two phases of the model. An example of the parameter space of the model, showing the critical line, can be seen in Figure 1, on page 22. To the right of this line the model is symmetric and we have $\langle \vec{\varphi}(n_\mu) \rangle = 0$. To the left, the symmetry is broken and we have $\langle \varphi_{\mathfrak{N}}(n_\mu) \rangle \neq 0$.

It might seem odd that we find here what looks like a complete phase transition even on finite lattices. In fact, it is a known fact that there are no real phase transitions on finite lattices with periodical boundary conditions, a situation in which all one can hope to get are approximations of this behavior. However, it is in fact possible to get complete phase transitions on finite lattices if one uses other boundary conditions or changes other

aspects of the system, such as the imposition of self-consistency conditions [2], just as we do in the Gaussian-Perturbative technique. Strictly speaking, however, the position of the critical line that we find here is not completely well-defined on finite lattices, because there is a slight dependence on α_0 and $\alpha_{\mathfrak{N}}$ through σ_0 and $\sigma_{\mathfrak{N}}$. This small dependence vanishes in the continuum limit, of course.

Since σ_0 and $\sigma_{\mathfrak{N}}$ are strictly positive, and $(\mathfrak{N} - 1) \geq 0$, we can see that this critical line starts at the Gaussian point $(\alpha, \lambda) = (0, 0)$, and extends to the quadrant where $\alpha < 0$ and $\lambda > 0$. Besides, since in the continuum limit σ_0 and $\sigma_{\mathfrak{N}}$ become identical, we may write the following equivalent equation for purposes of that limit,

$$\alpha + (\mathfrak{N} + 2)\lambda\sigma_0^2 = 0. \quad (8)$$

This is the known result for the critical line, obtained previously without the introduction of any external sources at all [1].

Going back to the general case, when the external source j_0 is not zero, then the equation of the critical line determines it in terms of v_0 , in either phase of the model, for as we saw before in Equation (4) we have

$$j_0 = v_0 \left\{ \lambda v_0^2 + \alpha + \lambda [(\mathfrak{N} - 1)\sigma_0^2 + 3\sigma_{\mathfrak{N}}^2] \right\}.$$

Given a point (α, λ) in the parameter space of the model, this clearly and directly determines j_0 in terms of v_0 . Conversely, given j_0 one may determine the corresponding v_0 by solving this simple algebraic cubic equation. Using the result for $\alpha_{\mathfrak{N}}$ in Equation (6), we may write this cubic equation in a simpler and more explicit form, in terms of the renormalized mass,

$$v_0^3 - \left(\frac{\alpha_{\mathfrak{N}}}{2\lambda} \right) v_0 + \left(\frac{j_0}{2\lambda} \right) = 0. \quad (9)$$

In the simple case in which we make $\lambda = 0$, returning to the free-field theory, we at once have that $\alpha_{\mathfrak{N}} = \alpha_0 = \alpha$, and the result reduces to

$$j_0 = \alpha v_0,$$

which is the familiar result for the free theory.

Transversal Mass: We are now in a position to discuss the physical situation of the transversal renormalized mass in the general case, in which j_0 and v_0 are not necessarily zero. This has to be done separately in each phase of the model, and taking explicitly in consideration whether or not there is a non-zero external source.

Symmetric Phase: In this case, if there is no external source, then we have $v_0 = 0$ and therefore from Equation (5) the renormalized mass parameter is given by

$$\alpha_0 = \alpha + \lambda [(\mathfrak{N} + 1)\sigma_0^2 + \sigma_{\mathfrak{N}}^2], \quad (10)$$

which is a positive quantity in this phase. On the other hand, if there is an external source j_0 , then there is also some value of v_0 associated to it, and therefore according to Equation (5) the renormalized mass parameter changes to

$$\alpha_0 = \lambda v_0^2 + \alpha + \lambda [(\mathfrak{N} + 1)\sigma_0^2 + \sigma_{\mathfrak{N}}^2].$$

This means that, given a point (α, λ) in the parameter space of the model, the renormalized mass increases with v_0 and thus with the external source. Note however that α_0 does not

depend directly on the external source, but on v_0 instead. This indicates that, in the case of localized external sources, the renormalized mass should depend both on the external source and on the relative position between the external source and the point of measurement of the mass.

Broken-Symmetric Phase: In this case, if there is no external source, then instead of zero we have for v_0 the non-trivial solution that we will denote here by $v_{0,\text{SSB}}$, which according to Equation (7) is given by

$$\lambda v_{0,\text{SSB}}^2 = -\alpha - \lambda [(\mathfrak{N} - 1)\sigma_0^2 + 3\sigma_{\mathfrak{N}}^2],$$

which is a positive quantity in this phase. Substituting this for the term λv_0^2 in Equation (5) we get for the transversal renormalized mass parameter

$$\alpha_0 = 2\lambda (\sigma_0^2 - \sigma_{\mathfrak{N}}^2). \quad (11)$$

Since σ_0^2 and $\sigma_{\mathfrak{N}}^2$ become identical in the continuum limit, this seems to indicate that α_0 tends to zero in the limit and thus corresponds to zero mass m_0 in that limit. However, α_0 always goes to zero in the continuum limit, and the fact that it does so is *not* enough to guarantee that m_0 is zero in the limit. Therefore, further analysis of the limit is necessary, which we will do later.

Going back to the case in which there is an external source j_0 , we see that v_0 will be somewhat larger than the solution $v_{0,\text{SSB}}$. In this case we may add and subtract $\lambda v_{0,\text{SSB}}^2$ in Equation (5) and therefore write α_0 as

$$\alpha_0 = \lambda (v_0^2 - v_{0,\text{SSB}}^2) + 2\lambda (\sigma_0^2 - \sigma_{\mathfrak{N}}^2), \quad (12)$$

showing once more that the mass increases with the variation of v_0 beyond its spontaneous symmetry-breaking value $v_{0,\text{SSB}}$, and hence that it increases with the introduction of the external source. This represents the variation of α_0 as a consequence of a variation of v_0 beyond its spontaneous symmetry breaking value. In terms of the mass m_0 this variation is not linear, but quadratic in nature.

Longitudinal Mass: Finally, we may now discuss the physical situation of the longitudinal renormalized mass in the case in which v_0 is not necessarily zero. This discussion proceeds in the same lines as the previous one. Once more this has to be done separately in each phase of the model, and taking in consideration whether or not there is a non-zero external source.

Symmetric Phase: In this case, if there is no external source, then we have $v_0 = 0$ and therefore from Equation (6) the renormalized mass parameter is given by

$$\alpha_{\mathfrak{N}} = \alpha + \lambda [(\mathfrak{N} - 1)\sigma_0^2 + 3\sigma_{\mathfrak{N}}^2],$$

which is a positive quantity in this phase. It is interesting to observe that, since in the continuum limit σ_0 and $\sigma_{\mathfrak{N}}$ become identical, for the purposes of that limit this equation is identical to the corresponding result for α_0 , shown in Equation (10), thus exhibiting the symmetry of the model in this phase. On the other hand, if there is an external source, then there is also some value of v_0 associated to it, and therefore according to Equation (6) the renormalized mass parameter changes to

$$\alpha_{\mathfrak{N}} = 3\lambda v_0^2 + \alpha + \lambda [(\mathfrak{N} - 1)\sigma_0^2 + 3\sigma_{\mathfrak{N}}^2] .$$

This means that, given a point (α, λ) in the parameter space of the model, the renormalized mass increases with v_0 and thus with the external source. This is now different from α_0 , since it increases three times as fast with v_0^2 . Note that once again $\alpha_{\mathfrak{N}}$ does not depend directly on the external source, but on v_0 instead.

Broken-Symmetric Phase: In this case, if there is no external source, then instead of zero we have for v_0 the non-trivial solution $v_{0,\text{SSB}}$, which according to Equation (7) is given by

$$\lambda v_{0,\text{SSB}}^2 = -\alpha - \lambda [(\mathfrak{N} - 1)\sigma_0^2 + 3\sigma_{\mathfrak{N}}^2] ,$$

which is a positive quantity in this phase. Substituting this for the term λv_0^2 in Equation (6) we get for the longitudinal renormalized mass parameter

$$\alpha_{\mathfrak{N}} = -2 \{ \alpha + \lambda [(\mathfrak{N} - 1)\sigma_0^2 + 3\sigma_{\mathfrak{N}}^2] \} . \quad (13)$$

This is a positive quantity in this phase, and in general corresponds to a non-zero mass $m_{\mathfrak{N}}$. If, however, there is an external source, then v_0 will be somewhat larger than the solution $v_{0,\text{SSB}}$. In this case we may add and subtract $3\lambda v_{0,\text{SSB}}^2$ to Equation (6) and therefore write $\alpha_{\mathfrak{N}}$ as

$$\alpha_{\mathfrak{N}} = 3\lambda (v_0^2 - v_{0,\text{SSB}}^2) - 2 \{ \alpha + \lambda [(\mathfrak{N} - 1)\sigma_0^2 + 3\sigma_{\mathfrak{N}}^2] \} ,$$

showing once more that the mass increases with the variation of v_0 beyond its spontaneous symmetry-breaking value, and hence that it increases with the introduction of the external source. If we denote the value of $\alpha_{\mathfrak{N}}$ without the presence of the source by $\alpha_{\mathfrak{N},\text{SSB}}$, we may write for the variation of $\alpha_{\mathfrak{N}}$ due to the external source

$$\alpha_{\mathfrak{N}} - \alpha_{\mathfrak{N},\text{SSB}} = 3\lambda (v_0^2 - v_{0,\text{SSB}}^2) . \quad (14)$$

Observe that once again this variation is three times larger than the corresponding variation of α_0 .

4.2 Continuum Limits

First of all, it is necessary to say that, regardless of the spacetime dimension and of the symmetry group which are chosen, the dimensionless parameters α and λ are the true free parameters of the model. Besides the requirements of stability, there is no reason to limit their range a priori. Limitations may arise, however, from the discussion of physically meaningful observables, expressed as expectation values, specially in the continuum limit. We start therefore with no more than the stability conditions that $\lambda \geq 0$, and that $\alpha \geq 0$ if $\lambda = 0$, as the limitations for α and λ .

In the continuum limit, when we make $N \rightarrow \infty$ and $a \rightarrow 0$, most dimensionless renormalized quantities we calculated here go to zero. In order to recover the physically meaningful results in the limit, before we take the limit we must rewrite these dimensionless quantities in terms of the corresponding dimensionfull quantities, using the scalings listed in Section 2, Equation (1). Since in the continuum limit σ_0 and $\sigma_{\mathfrak{N}}$ become identical, in all

cases where this is possible we will write the formulas in terms of σ_0 only, producing in this way equations which are equivalent to the original ones for the purposes of that limit.

Starting with the expectation value of the field, in the case in which there is no external source j_0 , in which case the limit must be taken within the broken-symmetric phase of the model if we are to have the possibility of a non-zero result, from Equation (7) we have

$$\begin{aligned}\langle\phi_{\mathfrak{N}}(x_\mu)\rangle &= V_0 \\ &= \frac{v_0}{a^{(d-2)/2}} \\ &= \frac{\sqrt{-[\alpha + \lambda(\mathfrak{N} + 2)\sigma_0^2]}}{\sqrt{\lambda} a^{(d-2)/2}}.\end{aligned}$$

Since for $d \geq 3$ the denominator goes to zero in the continuum limit, if the field $\phi_{\mathfrak{N}}(x_\mu)$ is to have a finite expectation value, then it is necessary that v_0 approach zero in the limit, which forcefully takes us to points over the critical line, which is characterized by $v_0 = 0$ and by the equation that states that the quantity within the square root above is zero.

Since the critical line starts at the Gaussian point and extends to the quadrant where $\lambda > 0$ and $\alpha \leq 0$, it follows that all possible continuum limits originating from the broken-symmetric phase must go to points in the parameter plane where $\alpha \leq 0$, the case $\alpha = 0$ being the Gaussian point and corresponding to the Gaussian sector of the model. In $d = 4$, in particular, *all* possible non-trivial continuum limits necessarily correspond to strictly negative values of α . A particular sequence of values of (α, λ) approaching the critical line defines both a path in the parameter space of the model and a rate of progress along that path, leading to that particular continuum limit, and is called a continuum limit *flow*. A continuum limit is completely characterized by its flow, and is *not* characterized completely just by a point (α, λ) in the parameter plane.

Going back to the case in which we have an external source present, we may now rewrite Equation (9) in terms of the renormalized dimensionfull quantities, thus obtaining

$$a^{d-4}V_0^3 - \left(\frac{m_{\mathfrak{N}}^2}{2\lambda}\right)V_0 + \left(\frac{J_0}{2\lambda}\right) = 0.$$

In the case $d = 3$ we see that, if λ is not zero, then the first term dominates over the others, and therefore we conclude simply that $V_0^3 = 0$. It follows that in this case there is no spontaneous symmetry breaking and no effect of the external source over V_0 in the continuum limit. If we wish to have any interesting structure in the model in this case, we are forced to make $\lambda = 0$ in the limit. If we do that at the appropriate rate, there may be interesting continuum limits sitting right over the Gaussian point. In the case $d \geq 5$, on the other hand, we see that the first term vanishes, and we are left with $J_0 = m_{\mathfrak{N}}^2 V_0$, which is characteristic of a free, or trivial theory. In the case $d = 4$ we get the equation

$$V_0^3 - \left(\frac{m_{\mathfrak{N}}^2}{2\lambda}\right)V_0 + \left(\frac{J_0}{2\lambda}\right) = 0.$$

It is interesting to calculate the discriminant Δ of this cubic equation, which turns out to be

$$\Delta = \frac{3^3}{2^2\lambda^2} \left(\sqrt{\frac{2}{3^3\lambda}} m_{\mathfrak{N}}^3 + J_0 \right) \left(\sqrt{\frac{2}{3^3\lambda}} m_{\mathfrak{N}}^3 - J_0 \right).$$

We can see now that the number of roots of the equation depends on the value of J_0 in a simple way. If we have

$$J_0 < \sqrt{\frac{2}{3^3\lambda}} m_{\mathfrak{N}}^3,$$

then $\Delta > 0$ and therefore there are three distinct simple real roots. If we have

$$J_0 = \sqrt{\frac{2}{3^3\lambda}} m_{\mathfrak{N}}^3,$$

then $\Delta = 0$ and the three roots merge into one triple real root. Finally, if

$$J_0 > \sqrt{\frac{2}{3^3\lambda}} m_{\mathfrak{N}}^3,$$

then $\Delta < 0$ and there is a single real root, the other two having non-zero imaginary parts. This supports the idea that as J_0 increases along positive values, the left well of the potential becomes shallower and eventually there is no possibility for the local distribution of the field $\varphi_{\mathfrak{N}}$ to fit within it, even to form a meta-stable state. One of the roots relates to the third extremum of the potential, the local maximum between the two minima. It is clear that, when there is more than one solution to the equation, only the largest solution corresponds to a stable state and is therefore relevant in the context of the symmetry breaking driven by a positive J_0 .

The same analysis regarding critical behavior and the critical line is valid for the renormalized masses. Considering first the limits from the symmetric phase, with no external source j_0 , we have $\alpha_0 = \alpha_{\mathfrak{N}}$ with $m_0^2 = \alpha_0/a^2$, and therefore using Equation (10) we have

$$\begin{aligned} \alpha_0 &= \alpha + \lambda(\mathfrak{N} + 2)\sigma_0^2 \Rightarrow \\ m_0 &= \frac{\sqrt{\alpha + \lambda(\mathfrak{N} + 2)\sigma_0^2}}{a}. \end{aligned}$$

We can see that, regardless of how we take the limit, we will necessarily have $m_0 = m_{\mathfrak{N}}$ in this case. Observe that the numerator on the right-hand side is the quantity which, according to the equation of the critical line, is zero over that line, and hence approaches zero when (α, λ) tends to a point on the critical line. Once more we see that, if we are to have a finite value for m_0 , we must approach the critical line on the continuum limit, in such a way that the quantity $\alpha + \lambda(\mathfrak{N} + 2)\sigma_0^2$ approaches zero as a^2 or faster. If the approach is such that the quantity in the numerator behaves exactly as a^2 , then we have a finite and non-zero value of m_0 . If the approach is faster than that, then we will have $m_0 = 0$. On the other hand, if it is too slow, then we may end up with an infinite m_0 in the limit.

The same type of mechanism works for limits from the broken-symmetric phase, except that in that case we will always have $m_0 = 0$ in the limit, as we will now demonstrate. As we saw before in Equation (11), we have for α_0

$$\alpha_0 = 2\lambda(\sigma_0^2 - \sigma_{\mathfrak{N}}^2),$$

which indeed goes to zero in the limit. However, the analysis of the limit is not so simple, due to the fact that on finite lattices α_0 appears in the right-hand side of the equation as well. If we write it explicitly, using Equations (B.6) and (B.7) of Appendix B, we get an equation involving α_0 and $\alpha_{\mathfrak{N}}$,

$$\begin{aligned}
\alpha_0 &= 2\lambda \frac{1}{N^d} \sum_{k_\mu}^{N^d} \left[\frac{1}{\rho^2(k_\mu) + \alpha_0} - \frac{1}{\rho^2(k_\mu) + \alpha_{\mathfrak{N}}} \right] \\
&= 2\lambda (\alpha_{\mathfrak{N}} - \alpha_0) \frac{1}{N^d} \sum_{k_\mu}^{N^d} \frac{1}{[\rho^2(k_\mu) + \alpha_0] [\rho^2(k_\mu) + \alpha_{\mathfrak{N}}]}.
\end{aligned}$$

Now, if $\alpha_{\mathfrak{N}} = \alpha_0$, which implies that $m_{\mathfrak{N}} = m_0$, then the right-hand side is zero, and therefore so is α_0 . This in turn implies that $m_0 = 0$, as expected. This is in fact one possibility, we may indeed have both m_0 and $m_{\mathfrak{N}}$ zero in the limit. If, on the other hand, we have $\alpha_{\mathfrak{N}} \neq \alpha_0$, then we may write the equation as

$$\frac{m_0^2}{m_{\mathfrak{N}}^2 - m_0^2} = 2\lambda \frac{1}{N^d} \sum_{k_\mu}^{N^d} \frac{1}{[\rho^2(k_\mu) + \alpha_0] [\rho^2(k_\mu) + \alpha_{\mathfrak{N}}]},$$

where we wrote the left-hand side in terms of dimensionfull quantities. Obviously, because both λ and the sum are necessarily positive quantities, it is impossible to have $m_{\mathfrak{N}} < m_0$. Here we see that, if we have both m_0 and $m_{\mathfrak{N}} > m_0$ different from zero in the limit, then the left-hand side has a non-zero limit and therefore the normalized sum on the right-hand side must be non-zero in the limit.

However, one can check numerically that, for $d \geq 4$ and in the type of continuum limit that we consider here, the normalized sum does indeed go to zero in the limit. This implies that in these dimensions, which include $d = 4$, we cannot have both m_0 and $m_{\mathfrak{N}}$ different from zero in the limit. Since $m_{\mathfrak{N}} > m_0$, this implies that we must always have $m_0 = 0$ in the limit. What we have here, as one should expect, are the Goldstone bosons brought about by the process of spontaneous symmetry breaking.

For the longitudinal mass parameter $m_{\mathfrak{N}}$ we have, using Equation (13),

$$\begin{aligned}
\alpha_{\mathfrak{N}} &= -2 [\alpha + \lambda(\mathfrak{N} + 2)\sigma_0^2] \Rightarrow \\
m_{\mathfrak{N}} &= \frac{\sqrt{-2 [\alpha + \lambda(\mathfrak{N} + 2)\sigma_0^2]}}{a},
\end{aligned}$$

so that exactly the same argument that was used for α_0 in the symmetric phase applies. We see therefore that the need to approach the critical line when one takes continuum limits in this model is a rather general characteristic of the model. This makes the critical line the locus of *all* physically possible continuum limits of the model. This means that making $\alpha < 0$ is not a choice that we have, since it is *forced* upon us by the need to obtain physically meaningful continuum limits.

Let us now discuss the continuum limits of the transversal renormalized mass in the presence of an external source. We have the result in Equation (5), valid in either phase,

$$\alpha_0(j_0) = \lambda v_0^2 + [\alpha + \lambda(\mathfrak{N} + 2)\sigma_0^2].$$

Observe that this equation implies that it is still necessary to approach the critical line in the continuum limit, and in the same ways as before. In the symmetric phase, if $\alpha_0(0)$ is the corresponding result in the absence of external sources, which corresponds to $v_0 = 0$, we may write

$$\alpha_0(j_0) = \alpha_0(0) + \lambda v_0^2.$$

Rewriting all quantities in terms of the corresponding dimensionfull ones we have

$$m_0^2(J_0) = m_0^2(0) + \lambda a^{d-4} V_0^2.$$

We see that for $d = 3$ we are forced to make $\lambda \rightarrow 0$ in the limit. In the case $d = 4$ no additional constraints on λ arise, and we get the relation

$$m_0(J_0) = \sqrt{m_0^2(0) + \lambda V_0^2},$$

describing indirectly how $m_0(J_0)$ increases with J_0 through the variation of V_0 . In the case $d = 5$ the term containing λ vanishes in the limit, and we get simply that $m_0(J_0) = m_0(0)$, meaning that in this case $m_0(J_0)$ does not really depend on J_0 in the continuum limit.

In the broken-symmetric phase we may start with Equation (12) for the transversal mass parameter. If we recall that we have already shown that in this phase we must have $m_0(0) = 0$ in the limit, we may make $\sigma_{\mathfrak{N}} = \sigma_0$ in this formula and thus obtain

$$\alpha_0 = \lambda (v_0^2 - v_{0,\text{SSB}}^2).$$

In terms of dimensionfull quantities we have therefore

$$m_0^2(J_0) = \lambda a^{d-4} (V_0^2 - V_{0,\text{SSB}}^2),$$

which gives us back $m_0(0) = 0$ in the absence of external sources. Not much changes in the discussion of the various possible dimensions. We may restrict our comments to the case $d = 4$, in which we get a fairly simple relation giving $m_0(J_0)$ in the presence of the external source,

$$m_0(J_0) = \sqrt{\lambda} \sqrt{V_0^2 - V_{0,\text{SSB}}^2}.$$

The same analysis can be made for the longitudinal mass in the presence of an external source. In this case we have the result in Equation (6), valid in either phase,

$$\alpha_{\mathfrak{N}}(j_0) = 3\lambda v_0^2 + [\alpha + \lambda(\mathfrak{N} + 2)\sigma_0^2].$$

The necessity to approach the critical line remains in force here. In the symmetric phase, if $\alpha_{\mathfrak{N}}(0)$ is the corresponding result in the absence of external sources, which corresponds to $v_0 = 0$, we may write

$$\alpha_{\mathfrak{N}}(j_0) = \alpha_{\mathfrak{N}}(0) + 3\lambda v_0^2.$$

Rewriting all quantities in terms of the corresponding dimensionfull ones we have

$$m_{\mathfrak{N}}^2(J_0) = m_{\mathfrak{N}}^2(0) + 3\lambda a^{d-4} V_0^2.$$

Once more we see that for $d = 3$ we are forced to make $\lambda \rightarrow 0$ in the limit. In the case $d = 4$ we get simply the relation

$$m_{\mathfrak{N}}(J_0) = \sqrt{m_{\mathfrak{N}}^2(0) + 3\lambda V_0^2},$$

describing indirectly how $m_{\mathfrak{N}}(J_0)$ increases with J_0 through the variation of V_0 . In the case $d = 5$ the term containing λ vanishes in the limit, and we get simply that $m_{\mathfrak{N}}(J_0) = m_{\mathfrak{N}}(0)$, meaning that in this case $m_{\mathfrak{N}}(J_0)$ also does not depend on J_0 in the continuum limit.

In the broken-symmetric phase we may start with Equation (14) for the longitudinal mass parameter

$$\alpha_{\mathfrak{N}} - \alpha_{\mathfrak{N},\text{SSB}} = 3\lambda (v_0^2 - v_{0,\text{SSB}}^2).$$

where $\alpha_{\mathfrak{N},\text{SSB}}$ is the value of the parameter in the absence of external sources. In terms of dimensionfull quantities we have therefore

$$m_{\mathfrak{N}}^2(J_0) - m_{\mathfrak{N}}^2(0) = 3\lambda a^{d-4} (V_0^2 - V_{0,\text{SSB}}^2),$$

Once again not much changes in the discussion of the various possible dimensions. In the case $d = 4$ we get

$$m_{\mathfrak{N}}(J_0) = \sqrt{m_{\mathfrak{N}}^2(0) + 3\lambda (V_0^2 - V_{0,\text{SSB}}^2)}.$$

It is interesting to note that, both for the transversal and longitudinal masses, the dependence of the renormalized masses on the external source J_0 seems to be a peculiar feature of the case $d = 4$, which is absent for $d \geq 5$.

5 Some Consequences

The calculations performed in this paper have a few rather interesting consequences, and some relevant conclusions can be drawn from them. In this section we discuss some of these consequences.

5.1 Triviality Tests

The fact that the renormalized masses depend on the external sources, as we saw above, has important consequences for the design of computer simulations targeted at probing the triviality issue. One way to do this is to perform simulations on finite lattices in which one tries to measure the relation between the external sources and the expectation value of the field. The argument is based on the fact that on symmetry grounds it is reasonable to expect that the model has an effective action with the general form

$$\begin{aligned} \Gamma_N[\vec{\varphi}_c] = & \sum_{n_\mu}^{N^d} \left\{ \frac{1}{2} \sum_{\nu}^d [\Delta_{\nu} \vec{\varphi}_c(n_\mu) \cdot \Delta_{\nu} \vec{\varphi}_c(n_\mu)] + \right. \\ & + \frac{\alpha_0}{2} [\vec{\varphi}_c(n_\mu) \cdot \vec{\varphi}_c(n_\mu)] + \frac{\alpha_{\mathfrak{N}} - \alpha_0}{2} \varphi_{\mathfrak{N},c}^2(n_\mu) + \\ & \left. + \frac{\lambda_R}{4} [\vec{\varphi}_c(n_\mu) \cdot \vec{\varphi}_c(n_\mu)]^2 - j_0 \varphi_{\mathfrak{N},c}(n_\mu) \right\}, \end{aligned}$$

where $\vec{\varphi}_c(n_\mu)$ is the classical field, which is just another name for the expectation value of the field, given an arbitrary external source j_0 . In this expression α_0 and $\alpha_{\mathfrak{N}}$ are the renormalized masses, and λ_R is the renormalized coupling constant. It is possible that additional terms may appear in $\Gamma_N[\vec{\varphi}_c]$, but terms containing derivatives are not relevant for the argument that follows, and terms with higher powers can be easily included in the analysis if need be.

If one considers only homogeneous external sources j_0 , then it is clear that the classical field must also be a constant, $\varphi_{\mathfrak{N},c}(n_\mu) = v_0$, and hence all terms containing derivatives

vanish. We are left with only the part of the effective action that contains the effective potential. In addition to this, since the external source is in the direction of $\varphi_{\mathfrak{N},c}(n_\mu)$ in the internal $SO(\mathfrak{N})$ space, it is clear that the expectation values of all the other field components are zero, so that we are left with

$$\Gamma_N[\vec{\varphi}_c] = \sum_{n_\mu}^{N^d} \left[\frac{\alpha_{\mathfrak{N}}}{2} v_0^2 + \frac{\lambda_R}{4} v_0^4 - j_0 v_0 \right],$$

where we now wrote v_0 for the expectation value of the field. Since the behavior of $\varphi_{\mathfrak{N},c}(n_\mu)$ is ruled by the minimum of this action in the classical or long-wavelength limit, which is consistent with the use of a homogeneous external source, we may now differentiate with respect to the classical field v_0 , and equate the result to zero, thus obtaining

$$j_0 = \alpha_{\mathfrak{N}} v_0 + \lambda_R v_0^3.$$

By measuring v_0 as a function of j_0 one may then determine from this equation the coefficients $\alpha_{\mathfrak{N}}$ and λ_R , and thus probe into the triviality of the model. For $d \geq 4$ triviality would result if it can be shown that

$$\lim_{N \rightarrow \infty} \lambda_R = 0.$$

In other words, a linear result for the relation between j_0 and v_0 , with the renormalized mass parameter as the coefficient, indicates triviality. Any deviations from linearity imply the existence of interactions, either on finite lattices or in the continuum limit. This technique avoids the necessity for the direct measurement of the four-point function, which is generally much more difficult to do numerically than to measure the one-point and two-point functions.

However, we have shown here that $\alpha_{\mathfrak{N}}$ itself depends on j_0 . Therefore, even if λ_R is in fact zero on finite lattices this relation will not result linear if the simulations are performed by varying j_0 at fixed values of parameters α and λ . It is therefore necessary to adjust these parameters, as one varies j_0 , in order to keep $\alpha_{\mathfrak{N}}$ constant. For this purpose the value of $\alpha_{\mathfrak{N}}$ can be obtained independently via the measurement of the propagator of the field component $\varphi_{\mathfrak{N}}(n_\mu)$, of course. At the end of the day, its value can be confirmed by the value resulting for the linear coefficient from a polynomial fit to the relation between j_0 and v_0 .

Given a certain chosen value for $\alpha_{\mathfrak{N}}$, for each value of j_0 one must search the parameter plane of the model looking for a point where $\alpha_{\mathfrak{N}}$ has that value, and only then measure v_0 . This can become a computationally expensive search. This can be done by keeping λ constant and varying α , thus traversing horizontal lines on the parameter plane, or by keeping α constant and varying λ , thus traversing vertical lines. Given the structure of the phase transition and of the critical line, one attractive alternative is to keep $\alpha^2 + \lambda^2$ constant and vary the angle θ around the position of the critical line, where

$$\frac{\lambda}{-\alpha} = \tan(\theta),$$

not forgetting that in general α will be negative. In any case, the formula giving $\alpha_{\mathfrak{N}}$ in terms of α , λ , v_0 and j_0 that we derived here, shown in Equation (6), may serve to provide at least a good initial guess for this costly search in the parameter plane.

5.2 Standard Model

The four-component $SO(4)$ model has an important application in the Standard Model of high-energy elementary particles. The field component $\varphi_{\mathfrak{N}}(n_\mu)$ corresponds in this case to the Higgs field. In this application the continuum limit must be taken from the broken-symmetric phase, for it is essential that we have, in the limit, a non-zero V_0 due to spontaneous symmetry breaking.

It is certainly possible to take limits from the broken-symmetric phase to the critical line in such a way that either V_0 or $m_{\mathfrak{N}}$ has a finite and non-zero limit. It is not so obvious, but true in $d = 4$, that one can take limits in which *both* are simultaneously finite and non-zero. In fact, the calculations imply that in this case there is a definite relation between V_0 and $m_{\mathfrak{N}}$.

If we recall our results for v_0 and $\alpha_{\mathfrak{N}}$ in the broken-symmetric phase (Equations (7) and (13)), without external sources, we have

$$\begin{aligned}\lambda v_0^2 &= -[\alpha + \lambda(\mathfrak{N} + 2)\sigma_0^2], \\ \alpha_{\mathfrak{N}} &= -2[\alpha + \lambda(\mathfrak{N} + 2)\sigma_0^2].\end{aligned}$$

It immediately follows that we have the following result relating v_0 and $\alpha_{\mathfrak{N}}$,

$$2\lambda v_0^2 = \alpha_{\mathfrak{N}}.$$

Writing this in terms of dimensionfull quantities we get

$$2\lambda a^{d-4} V_0^2 = m_{\mathfrak{N}}^2.$$

Of course the important dimension here is $d = 4$, but let us comment on the other cases anyway. In $d = 3$ we are forced once again to make $\lambda \rightarrow 0$, which takes us to the Gaussian point, and if we do this at the appropriate pace, we then simply get $2\lambda V_0^2 = m_{\mathfrak{N}}^2$. In $d = 5$ we conclude that, so long as λ and V_0 are finite, we must have $m_{\mathfrak{N}} = 0$. If we insist on a finite and non-zero $m_{\mathfrak{N}}$, then V_0 must diverge to infinity. So in this case we cannot take a limit in such a way that both V_0 and $m_{\mathfrak{N}}$ remain finite and non-zero.

However, in $d = 4$, and only in $d = 4$, we get a definite relation between V_0 and $m_{\mathfrak{N}}$, involving only the dimensionless parameters of the model, and valid for all allowed values of these parameters within the broken-symmetric phase, given by

$$\frac{V_0}{m_{\mathfrak{N}}} = \frac{1}{\sqrt{2\lambda}}.$$

Since the values of V_0 and $m_{\mathfrak{N}}$ are known experimentally, namely $V_0 \approx 246$ Gev and $m_{\mathfrak{N}} \approx 126$ Gev, we immediately get a result for λ ,

$$\lambda \approx 0.131.$$

Given this result, we can find α as well. All we have to do is to use the equation of the critical line, given in Equation (8),

$$\alpha + \lambda(\mathfrak{N} + 2)\sigma_0^2 = 0,$$

with $\mathfrak{N} = 4$ and our best numerical evaluation of σ_0^2 for $d = 4$, which is $\sigma_0^2 \approx 0.15493$, and we get

$$\alpha \approx -0.122.$$

Conceptually, this is a rather remarkable result. Please observe that we are not using the experimental data to make statements about expectation values, but instead to determine the values of bare dimensionless parameters within the mathematical structure of the model. We are able, using the experimental data, to pinpoint the location in the parameter space of the model, along the critical line, where it must be located if it is applicable to the real world,

$$(\alpha, \lambda) \approx (-0.122, 0.131).$$

This is a point at a distance of approximately 0.179 from the Gaussian point, along the critical line, which makes an angle of approximately 47.0 degrees with the negative α semi-axis. The situation in the parameter-plane of the model is depicted in Figure 1, which is drawn approximately to scale.

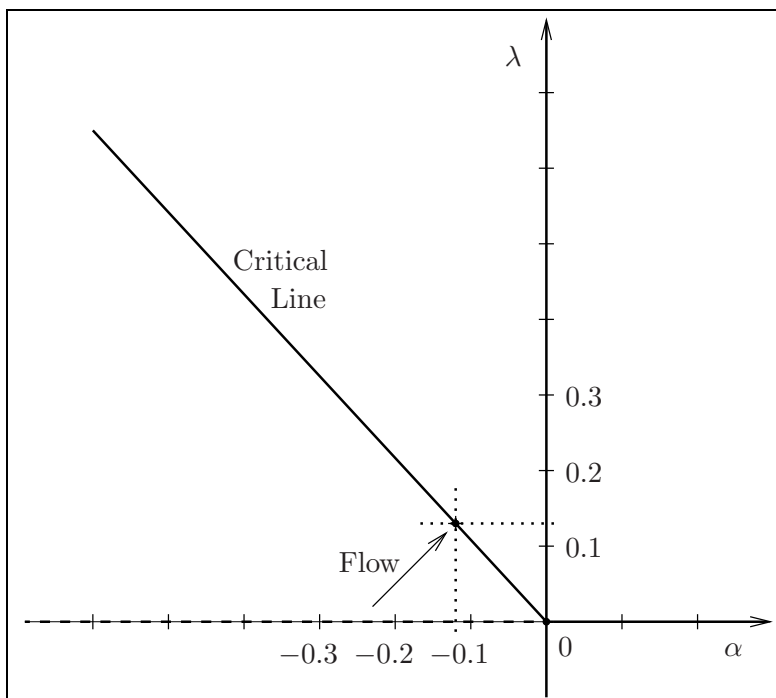


Figure 1: Critical diagram of the model in $d = 4$, for $\mathfrak{N} = 4$, with the Standard Model continuum limit point $(\alpha, \lambda) \approx (-0.122, 0.131)$ singled out, showing the path of a possible continuum limit flow.

One may wonder how accurate this result may be. In the Standard Model there are electroweak charges associated to $\vec{\varphi}(n_\mu)$, which are being ignored here. It is of course possible that these other interactions might change the expectation value and the renormalized mass of the Higgs field. However, after the symmetry is broken and the three Goldstone bosons $\varphi_i(n_\mu)$, $i = 1, 2, 3$ are absorbed by the three massive vector bosons, the single remaining scalar field which is the Higgs has no electromagnetic charge, and undergoes only weak interactions, if any. Therefore it is reasonable to think that whatever corrections there may be to the result above are probably quite small. By comparison to possible weak perturbative corrections, the results presented here have a rather brutal character, since they handle correctly the non-perturbative phenomenon of spontaneous symmetry breaking, at the quantum level, flipping the sign of α to negative values in that process.

5.3 Hints on Triviality

Since we have results for both v_0 and $m_{\mathfrak{N}}$ as functions of j_0 , it is conceivable that these results can give us some hints as to the question of triviality. As we shall see, trying to do this does in fact provide some rather crude hints, but most of all it puts in evidence the limitations of the calculational technique.

Our Gaussian-Perturbative result for the relation between j_0 and v_0 in either phase of the model, as shown in Equation (4), may be written for the purposes of the continuum limit as

$$j_0 = v_0 \left\{ \lambda v_0^2 + [\alpha + \lambda(\mathfrak{N} + 2)\sigma_0^2] \right\}.$$

On the other hand, our result for the renormalized mass parameter $\alpha_{\mathfrak{N}}$, also valid in either phase of the model, in the presence of the external source, as shown in Equation (6), may be written for the purposes of the continuum limit as

$$\alpha_{\mathfrak{N}} = 3\lambda v_0^2 + [\alpha + \lambda(\mathfrak{N} + 2)\sigma_0^2].$$

It follows that we may combine these two results, and write a relation involving the renormalized quantities v_0 and $\alpha_{\mathfrak{N}}$,

$$j_0 = v_0 (\alpha_{\mathfrak{N}} - 2\lambda v_0^2).$$

This relation is valid for all $d \geq 3$, for all $\mathfrak{N} \geq 1$, and all explicit references to α are gone. In the continuum limit both sides of this equation approach zero. In order to analyze the limit, it is necessary to rewrite everything in terms of the corresponding finite and possibly non-zero dimensionfull quantities. Doing this with the use of the scalings given in Section 2, Equation (1), we get

$$J_0 = V_0 \left(m_{\mathfrak{N}}^2 - 2\lambda a^{d-4} V_0^2 \right).$$

This behaves differently in each dimension d . We will analyze separately the cases $d = 3$, $d = 4$ and $d \geq 5$.

Case $d = 3$: The result becomes inconsistent unless we make $\lambda \rightarrow 0$ in the continuum limit, which takes us to the Gaussian point. If we do that sufficiently fast then we get $J_0 = m_{\mathfrak{N}}^2 V_0$, a result consistent with a linear theory. Otherwise, if we take λ to zero at the appropriate rate, we get

$$J_0 = V_0 (m_{\mathfrak{N}}^2 - 2\Lambda V_0^2).$$

Since the sign of the second term is reversed, this result does not seem to make much sense, even if we consider that the Λ that appears there is a bare parameter, a parameter characterizing a continuum-limit flow, in fact, and *not* the renormalized coupling constant.

Observe however that this is not necessarily a free theory, even at the Gaussian point, where one would expect that $\lambda_R = 0$. This is so because in $d = 3$ we have $\Lambda_R = \lambda_R/a$, and therefore we may have $\Lambda_R \neq 0$ even if $\lambda_R \rightarrow 0$ as we take the limit and make $a \rightarrow 0$. In other words, in $d = 3$ we may have interacting limits of this type sitting right on top of the Gaussian point.

The complete failure of the approximation away from the Gaussian point suggests that in that case the distribution of the $d = 3$ model may not be sufficiently close to a Gaussian distribution to allow a Gaussian approximation to work, and hence its expectations values cannot be well represented by the Gaussian measure of $S_0[\varphi]$.

Case $d = 4$: The result is perfectly regular, and is simply given by

$$J_0 = V_0 (m_{\mathfrak{N}}^2 - 2\lambda V_0^2) .$$

Not much can be concluded in this case, though. Of course we must not forget that both V_0 and $m_{\mathfrak{N}}$ are functions of λ and J_0 , in such a way that the right-hand side of this equation remains positive for positive J_0 .

Although this equation has the general form expected for an interacting theory, it is crucial to note that the sign of the second term is reversed. Since we must have $\lambda > 0$, this term is necessarily negative. This is not really all that surprising, for one must not forget that it is not to be expected that this calculational technique can produce predictions about λ_R , which is a parameter related to the fourth moment of the distribution, that is of course absent from a Gaussian approximation. The reversed sign, that seems to indicate that increasing λ works in the way opposite to what one would expect, appears there as a warning about this limitation. Of course, interpreting this term as a prediction for λ_R would be absurd, since it would imply that the renormalized coupling constant is negative, and thus would correspond to an unstable renormalized model.

Case $d \geq 5$: The result is not only perfectly regular, but the second term in the right-hand side vanishes in the limit, so long as λ is kept finite, and one is left with the simple result $J_0 = m_{\mathfrak{N}}^2 V_0$, which is consistent with a trivial theory. In this case this result is valid for any finite value of λ . This is consistent with the triviality of the model for $d \geq 5$, which seems to be a fairly well-established fact.

It is interesting to observe that it is possible to define a version of this model in which the limit $\lambda \rightarrow \infty$ is taken. It is possible to show, with all mathematical rigor, and for any d and any \mathfrak{N} , that the limit of the $SO(\mathfrak{N})$ polynomial model we have here, in which one makes $\lambda \rightarrow \infty$ and $\alpha \rightarrow -\infty$ in such a way that $\beta = -\alpha/\lambda$ is kept finite, is in fact the $SO(\mathfrak{N})$ non-linear Sigma Model with coupling constant β . It is therefore possible that the $SO(\mathfrak{N})$ non-linear Sigma Models in $d = 5$ or more may still have some interesting continuum limits.

6 Conclusions

It was already known, for some time, that the approximation scheme that we name here the Gaussian-Perturbative approximation gives good results for the $SO(\mathfrak{N})$ -symmetric $\lambda\phi^4$ model in $d = 4$, regarding its critical behavior [1]. It is interesting to speculate that the triviality of the model in $d = 4$, which is fairly well established numerically but still lacks rigorous proof, is somehow behind the fact that this approximation works as well as it does in that case. This is so because a trivial model would have a Gaussian effective action, which would allow for a good approximation of its expectation values by a Gaussian measure, which is what we do in the Gaussian-Perturbative approximation.

In this work we extended that technique to the same model in the presence of an external source. This resulted in specific predictions for the values of the expectation value of the field and for the renormalized masses as functions of the external source. Such predictions could motivate future numerical studies with the objective of evaluating their worth by comparing them to the results of appropriate stochastic simulations. In particular, the fact that the renormalized masses do depend on the external sources through the expectation value of the field is important for the very design of some such numerical simulations.

The simulations done in the past to test this technique used what we named back-rotation simulations, which introduce some additional uncertainties into the whole analysis.

This was done because neither in the analytical calculations nor in the numerical simulations we were capable at that time to deal appropriately with the external sources. It is now possible to perform simulations in the presence external sources, without the use of the back-rotation idea. With such simulations and the results presented in this paper, it should be possible to do much better comparison of the numerical and analytical results.

We also pointed out a simple and interesting consequence of the results regarding the application of the $\lambda\phi^4$ model in the Standard Model of particle physics. The results allow us to determine the critical point (α, λ) in the parameter plane of the model that should correspond to the continuum-limit flows leading to the Standard Model. This is a rather unique situation, in which actual experimental data is used to determine the values of bare, dimensionless parameters within the mathematical structure of the $\lambda\phi^4$ model.

Although this result in itself may be no more than a curiosity, it would be interesting to determine whether or not this technique and its results could not find a more widespread application for the computation of physical predictions from the Standard Model. This, combined with the use of the very probable fact that the renormalized coupling constant λ_R is in fact exactly zero in the continuum limits of this model, could very well result in the extraction of new and interesting insights from the Standard Model.

As an example, we may point out that the attribution of a negative value to the bare parameter α is not really a matter of choice, as is implied in the usual treatment of the Standard Model. It is in fact a very *strict requirement* for the existence of physically meaningful continuum limits of the model, as we have shown in this paper. There are in fact *no* continuum limits, in which the fundamental action is not Gaussian and the model has finite renormalized masses, such that $\alpha > 0$ in the limit.

Is in important to point out that the results presented here for critical behavior and symmetry breaking within the $\lambda\phi^4$ model are quite independent of the renormalized coupling constant λ_R . In particular, they are quite independent of whether or not λ_R is zero in the continuum limit. In other words, the probable triviality of the model in the continuum limit does not disturb the mechanism of phase transition and symmetry breaking, and hence would not void the Higgs mechanism.

References

- [1] “Symmetry breaking on a finite euclidean lattice”, J. L. deLyra and A. C. R. Martins, Nucl. Phys. **B432** (1994) 621-640.
- [2] “Finite lattice systems with true critical behavior”, J. L. deLyra, T. E. Gallivan and S.-K. Foong, Phys. Rev. **D46** (1992) 1643–1657.

A Calculation of Expectation Values

In this section we calculate in detail the expectation values which are needed for the evaluation of the observables discussed in this paper. These are all expectation values in the measure of the Gaussian action given in Equation (2),

$$S_0[\vec{\varphi}'] = \sum_{n_\mu}^{N^d} \left\{ \frac{1}{2} \sum_{\nu}^d [\Delta_{\nu} \vec{\varphi}'(n_{\mu}) \cdot \Delta_{\nu} \vec{\varphi}'(n_{\mu})] + \right. \\ \left. + \frac{\alpha_0}{2} [\vec{\varphi}'(n_{\mu}) \cdot \vec{\varphi}'(n_{\mu})] + \frac{\alpha_{\mathfrak{N}} - \alpha_0}{2} \varphi_{\mathfrak{N}}'^2(n_{\mu}) \right\},$$

which is even on the fields. They will all involve the non-Gaussian or “interacting” part of the action, which is given in Equation (3),

$$\begin{aligned}
S_V[\vec{\varphi}] = \sum_{n_\mu}^{N^d} & \left\{ v_0 [\alpha + \lambda v_0^2] \varphi'_{\mathfrak{N}}(n_\mu) - j_0 \varphi'_{\mathfrak{N}}(n_\mu) + \right. \\
& + \frac{\alpha - \alpha_0 + \lambda v_0^2}{2} [\vec{\varphi}'(n_\mu) \cdot \vec{\varphi}'(n_\mu)] + \frac{\alpha_0 - \alpha_{\mathfrak{N}} + 2\lambda v_0^2}{2} \varphi'^2_{\mathfrak{N}}(n_\mu) + \\
& \left. + \lambda v_0 [\vec{\varphi}'(n_\mu) \cdot \vec{\varphi}'(n_\mu)] \varphi'_{\mathfrak{N}}(n_\mu) + \frac{\lambda}{4} [\vec{\varphi}'(n_\mu) \cdot \vec{\varphi}'(n_\mu)]^2 \right\}.
\end{aligned}$$

There are field-odd and field-even terms in this action. Since the expectation values will single out one of these parities, it is convenient to write explicitly the field-odd and field-even parts of the non-Gaussian part of the action,

$$\begin{aligned}
S_{V,\text{odd}}[\vec{\varphi}] &= \sum_{n_\mu}^{N^d} \left\{ v_0 [\alpha + \lambda v_0^2] \varphi'_{\mathfrak{N}}(n_\mu) - j_0 \varphi'_{\mathfrak{N}}(n_\mu) + \right. \\
& \quad \left. + \lambda v_0 [\vec{\varphi}'(n_\mu) \cdot \vec{\varphi}'(n_\mu)] \varphi'_{\mathfrak{N}}(n_\mu) \right\}, \\
S_{V,\text{even}}[\vec{\varphi}] &= \sum_{n_\mu}^{N^d} \left\{ \frac{\alpha - \alpha_0 + \lambda v_0^2}{2} [\vec{\varphi}'(n_\mu) \cdot \vec{\varphi}'(n_\mu)] + \right. \\
& \quad + \frac{\alpha_0 - \alpha_{\mathfrak{N}} + 2\lambda v_0^2}{2} \varphi'^2_{\mathfrak{N}}(n_\mu) + \\
& \quad \left. + \frac{\lambda}{4} [\vec{\varphi}'(n_\mu) \cdot \vec{\varphi}'(n_\mu)]^2 \right\}.
\end{aligned}$$

It is also convenient, for use in the calculations, to write versions of these expressions in which the terms containing the $\varphi'_{\mathfrak{N}}(n_\mu)$ field component are written explicitly, and one version in which the terms containing the $\varphi'_1(n_\mu)$ field component are written explicitly as well, rather than as part of the scalar products,

$$\begin{aligned}
S_{V,\text{odd}}[\vec{\varphi}] &= \sum_{n_\mu}^{N^d} \left\{ v_0 [\alpha + \lambda v_0^2] \varphi'_{\mathfrak{N}}(n_\mu) - j_0 \varphi'_{\mathfrak{N}}(n_\mu) + \right. \\
& \quad \left. + \lambda v_0 \left[\sum_{i=1}^{\mathfrak{N}-1} \varphi'^2_i(n_\mu) \right] \varphi'_{\mathfrak{N}}(n_\mu) + \lambda v_0 \varphi'^3_{\mathfrak{N}}(n_\mu) \right\}, \tag{A.1}
\end{aligned}$$

$$\begin{aligned}
S_{V,\text{even}}[\vec{\varphi}] &= \sum_{n_\mu}^{N^d} \left\{ \frac{\alpha - \alpha_0 + \lambda v_0^2}{2} \sum_{i=1}^{\mathfrak{N}-1} \varphi'^2_i(n_\mu) + \right. \\
& \quad + \frac{\alpha - \alpha_{\mathfrak{N}} + 3\lambda v_0^2}{2} \varphi'^2_{\mathfrak{N}}(n_\mu) + \\
& \quad \left. + \frac{\lambda}{4} \left[\sum_{i=1}^{\mathfrak{N}-1} \varphi'^2_i(n_\mu) \right]^2 + \frac{\lambda}{2} \left[\sum_{i=1}^{\mathfrak{N}-1} \varphi'^2_i(n_\mu) \right] \varphi'^2_{\mathfrak{N}}(n_\mu) + \frac{\lambda}{4} \varphi'^4_{\mathfrak{N}}(n_\mu) \right\}, \tag{A.2}
\end{aligned}$$

$$S_{V,\text{even}}[\vec{\varphi}] = \sum_{n_\mu}^{N^d} \left\{ \frac{\alpha - \alpha_0 + \lambda v_0^2}{2} \varphi'^2_1(n_\mu) + \right.$$

$$\begin{aligned}
& + \frac{\alpha - \alpha_0 + \lambda v_0^2}{2} \sum_{i=2}^{\mathfrak{N}-1} \varphi_i'^2(n_\mu) + \\
& + \frac{\alpha - \alpha_{\mathfrak{N}} + 3\lambda v_0^2}{2} \varphi_{\mathfrak{N}}'^2(n_\mu) + \\
& + \frac{\lambda}{4} \varphi_1'^4(n_\mu) + \frac{\lambda}{2} \varphi_1'^2(n_\mu) \left[\sum_{i=2}^{\mathfrak{N}-1} \varphi_i'^2(n_\mu) \right] + \frac{\lambda}{4} \left[\sum_{i=2}^{\mathfrak{N}-1} \varphi_i'^2(n_\mu) \right]^2 + \\
& + \frac{\lambda}{2} \varphi_1'^2(n_\mu) \varphi_{\mathfrak{N}}'^2(n_\mu) + \frac{\lambda}{2} \left[\sum_{i=2}^{\mathfrak{N}-1} \varphi_i'^2(n_\mu) \right] \varphi_{\mathfrak{N}}'^2(n_\mu) + \frac{\lambda}{4} \varphi_{\mathfrak{N}}'^4(n_\mu) \Bigg\}.
\end{aligned} \tag{A.3}$$

A.1 The Expectation Value of $\varphi'_{\mathfrak{N}}(n'_\mu) S_V[\vec{\varphi}']$

Let us now calculate the expectation value

$$\langle \varphi'_{\mathfrak{N}}(n'_\mu) S_V[\vec{\varphi}'] \rangle_0.$$

Since $S_0[\vec{\varphi}']$ is field-even, all expectation values of field-odd observables are zero when calculated in its measure. Therefore it is necessary that the observables be field-even if their expectation values are to be non-zero. Since in this case we have an explicit factor of $\varphi'_{\mathfrak{N}}(n'_\mu)$, it follows that only the field-odd part of $S_V[\vec{\varphi}']$ will contribute to this expectation value,

$$\langle \varphi'_{\mathfrak{N}}(n'_\mu) S_V[\vec{\varphi}'] \rangle_0 = \langle \varphi'_{\mathfrak{N}}(n'_\mu) S_{V,\text{odd}}[\vec{\varphi}'] \rangle_0.$$

If we write the expectation value out, using the form of the action $S_{V,\text{odd}}[\vec{\varphi}']$ given in Equation (A.1), we get

$$\begin{aligned}
\langle \varphi'_{\mathfrak{N}}(n'_\mu) S_V[\vec{\varphi}'] \rangle_0 &= \sum_{n_\mu}^{N^d} \left\{ v_0 [\alpha + \lambda v_0^2] \langle \varphi'_{\mathfrak{N}}(n_\mu) \varphi'_{\mathfrak{N}}(n'_\mu) \rangle_0 + \right. \\
& - j_0 \langle \varphi'_{\mathfrak{N}}(n_\mu) \varphi'_{\mathfrak{N}}(n'_\mu) \rangle_0 + \\
& + \lambda v_0 \sum_{i=1}^{\mathfrak{N}-1} \langle \varphi_i'^2(n_\mu) \rangle_0 \langle \varphi'_{\mathfrak{N}}(n_\mu) \varphi'_{\mathfrak{N}}(n'_\mu) \rangle_0 + \\
& \left. + \lambda v_0 \langle \varphi_{\mathfrak{N}}'^3(n_\mu) \varphi'_{\mathfrak{N}}(n'_\mu) \rangle_0 \right\}.
\end{aligned}$$

The expectation values in the first three terms turn out to be just the position-space propagator for the $\varphi'_{\mathfrak{N}}(n_\mu)$ field component. From Appendix B, Equation (B.2), we get

$$g_{\mathfrak{N}}(n_\mu - n'_\mu) = \frac{1}{N^d} \sum_{k_\mu}^{N^d} \frac{e^{-\mathbf{i}(2\pi/N) \sum_\mu k_\mu (n_\mu - n'_\mu)}}{\rho^2(k_\mu) + \alpha_{\mathfrak{N}}},$$

which is just the statement that $g_{\mathfrak{N}}(n_\mu - n'_\mu)$ is the inverse Fourier transform of the momentum-space propagator. We also have the corresponding result for the other field $\mathfrak{N} - 1$ components, with $i \neq \mathfrak{N}$,

$$\begin{aligned}
g_0(n_\mu - n'_\mu) &= \langle \varphi'_i(n_\mu) \varphi'_i(n'_\mu) \rangle_0 \\
&= \frac{1}{N^d} \sum_{k_\mu} \frac{e^{-\mathbf{i}(2\pi/N) \sum_\mu k_\mu (n_\mu - n'_\mu)}}{\rho^2(k_\mu) + \alpha_0}.
\end{aligned}$$

The second expectation value that we must calculate, with $i \neq \mathfrak{N}$, is simply

$$\begin{aligned}
g_0(0) &= \langle \varphi_i'^2(n_\mu) \rangle_0 \\
&= \sigma_0^2 \\
&= \frac{1}{N^d} \sum_{k_\mu} \frac{1}{\rho^2(k_\mu) + \alpha_0}.
\end{aligned}$$

The third expectation value that we must calculate can be found in Appendix B, Equation (B.9), and can be shown to be given in terms of the first one by

$$\langle \varphi_{\mathfrak{N}}'^3(n_\mu) \varphi_{\mathfrak{N}}'(n'_\mu) \rangle_0 = 3\sigma_{\mathfrak{N}}^2 g_{\mathfrak{N}}(n_\mu - n'_\mu).$$

We are thus left with a simpler form for the expectation value,

$$\begin{aligned}
\langle \varphi_{\mathfrak{N}}'(n'_\mu) S_V[\vec{\varphi}'] \rangle_0 &= \sum_{n_\mu}^{N^d} \left\{ v_0 \alpha g_{\mathfrak{N}}(n_\mu - n'_\mu) + \right. \\
&\quad + v_0^3 \lambda g_{\mathfrak{N}}(n_\mu - n'_\mu) + \\
&\quad + v_0 (\mathfrak{N} - 1) \lambda \sigma_0^2 g_{\mathfrak{N}}(n_\mu - n'_\mu) + \\
&\quad + v_0 3 \lambda \sigma_{\mathfrak{N}}^2 g_{\mathfrak{N}}(n_\mu - n'_\mu) + \\
&\quad \left. - j_0 g_{\mathfrak{N}}(n_\mu - n'_\mu) \right\}.
\end{aligned}$$

In all terms the only quantity still depending on n_μ is $g_{\mathfrak{N}}(n_\mu - n'_\mu)$, so that we can write this equation as

$$\langle \varphi_{\mathfrak{N}}'(n'_\mu) S_V[\vec{\varphi}'] \rangle_0 = \{ v_0 [\alpha + v_0^2 \lambda + (\mathfrak{N} - 1) \lambda \sigma_0^2 + 3 \lambda \sigma_{\mathfrak{N}}^2] - j_0 \} \sum_{n_\mu}^{N^d} g_{\mathfrak{N}}(n_\mu - n'_\mu).$$

Using Equation (B.3), which gives this final sum, we may finally write

$$\langle \varphi_{\mathfrak{N}}'(n'_\mu) S_V[\vec{\varphi}'] \rangle_0 = \frac{v_0 [\alpha + v_0^2 \lambda + (\mathfrak{N} - 1) \lambda \sigma_0^2 + 3 \lambda \sigma_{\mathfrak{N}}^2] - j_0}{\alpha_{\mathfrak{N}}}. \quad (\text{A.4})$$

A.2 The Expectation Value of $S_V[\vec{\varphi}']$

We now calculate the expectation value

$$\langle S_V[\vec{\varphi}'] \rangle_0.$$

Only the field-even part of the action will yield a non-zero result, so that we have

$$\langle S_V[\vec{\varphi}'] \rangle_0 = \langle S_{V,\text{even}}[\vec{\varphi}'] \rangle_0.$$

Using the form of $S_{V,\text{even}}[\vec{\varphi}']$ shown in Equation (A.2) we get for this expectation value

$$\begin{aligned} \langle S_V[\vec{\varphi}'] \rangle_0 &= \sum_{n_\mu}^{N^d} \left\{ \frac{\alpha - \alpha_0 + \lambda v_0^2}{2} \sum_{i=1}^{\mathfrak{N}-1} \langle \varphi_i'^2(n_\mu) \rangle_0 + \right. \\ &\quad \left. + \frac{\alpha - \alpha_{\mathfrak{N}} + 3\lambda v_0^2}{2} \langle \varphi_{\mathfrak{N}}'^2(n_\mu) \rangle_0 + \right. \\ &\quad \left. + \frac{\lambda}{4} \left\langle \left[\sum_{i=1}^{\mathfrak{N}-1} \varphi_i'^2(n_\mu) \right]^2 \right\rangle_0 + \frac{\lambda}{2} \left[\sum_{i=1}^{\mathfrak{N}-1} \langle \varphi_i'^2(n_\mu) \rangle_0 \right] \langle \varphi_{\mathfrak{N}}'^2(n_\mu) \rangle_0 + \frac{\lambda}{4} \langle \varphi_{\mathfrak{N}}'^4(n_\mu) \rangle_0 \right\}. \end{aligned}$$

Most of the remaining expectation values can be written in terms of σ_0 and $\sigma_{\mathfrak{N}}$, if we recall that it can be shown that for $i \neq \mathfrak{N}$ we have

$$\langle \varphi_i'^4(n_\mu) \rangle_0 = 3\sigma_0^2,$$

while for $i = \mathfrak{N}$ we have, in a similar way,

$$\langle \varphi_{\mathfrak{N}}'^4(n_\mu) \rangle_0 = 3\sigma_{\mathfrak{N}}^2,$$

as one can find in Appendix B, Equation (B.8). Given all this, we may write for our expectation value

$$\begin{aligned} \langle S_V[\vec{\varphi}'] \rangle_0 &= \sum_{n_\mu}^{N^d} \left\{ \frac{\alpha - \alpha_0 + \lambda v_0^2}{2} (\mathfrak{N} - 1) \sigma_0^2 + \frac{\alpha - \alpha_{\mathfrak{N}} + 3\lambda v_0^2}{2} \sigma_{\mathfrak{N}}^2 + \right. \\ &\quad \left. + \frac{\lambda}{4} \left\langle \left[\sum_{i=1}^{\mathfrak{N}-1} \varphi_i'^2(n_\mu) \right]^2 \right\rangle_0 + \frac{\lambda}{2} (\mathfrak{N} - 1) \sigma_0^2 \sigma_{\mathfrak{N}}^2 + \frac{3\lambda}{4} \sigma_{\mathfrak{N}}^4 \right\}. \end{aligned}$$

The remaining expectation value of the sum shown can be found in Appendix B, Equation (B.15),

$$\left\langle \left[\sum_{i=1}^{\mathfrak{N}-1} \varphi_i'^2(n_\mu) \right]^2 \right\rangle_0 = (\mathfrak{N} + 1)(\mathfrak{N} - 1) \sigma_0^4.$$

Using this result we get for our expectation value

$$\begin{aligned} \langle S_V[\vec{\varphi}'] \rangle_0 &= \sum_{n_\mu}^{N^d} \left[\frac{\alpha - \alpha_0 + \lambda v_0^2}{2} (\mathfrak{N} - 1) \sigma_0^2 + \frac{\alpha - \alpha_{\mathfrak{N}} + 3\lambda v_0^2}{2} \sigma_{\mathfrak{N}}^2 + \right. \\ &\quad \left. + \frac{\lambda}{4} (\mathfrak{N}^2 - 1) \sigma_0^4 + \frac{\lambda}{2} (\mathfrak{N} - 1) \sigma_0^2 \sigma_{\mathfrak{N}}^2 + \frac{3\lambda}{4} \sigma_{\mathfrak{N}}^4 \right]. \end{aligned}$$

Note that all the sums can now be done, so that we can write our result in the simpler form

$$\begin{aligned} \langle S_V[\vec{\varphi}'] \rangle_0 &= N^d \left[\frac{\alpha - \alpha_0 + \lambda v_0^2}{2} (\mathfrak{N} - 1) \sigma_0^2 + \frac{\alpha - \alpha_{\mathfrak{N}} + 3\lambda v_0^2}{2} \sigma_{\mathfrak{N}}^2 + \right. \\ &\quad \left. + \frac{\lambda}{4} (\mathfrak{N}^2 - 1) \sigma_0^4 + \frac{\lambda}{2} (\mathfrak{N} - 1) \sigma_0^2 \sigma_{\mathfrak{N}}^2 + \frac{3\lambda}{4} \sigma_{\mathfrak{N}}^4 \right]. \end{aligned} \quad (\text{A.5})$$

A.3 The Expectation Value of $\varphi'_1(n'_\mu)\varphi'_1(n''_\mu)S_V[\vec{\varphi}']$

We now calculate the expectation value

$$\langle \varphi'_1(n'_\mu)\varphi'_1(n''_\mu)S_V[\vec{\varphi}'] \rangle_0.$$

Once more only the field-even part of the action will yield a non-zero result, so that we have

$$\langle \varphi'_1(n'_\mu)\varphi'_1(n''_\mu)S_V[\vec{\varphi}'] \rangle_0 = \langle \varphi'_1(n'_\mu)\varphi'_1(n''_\mu)S_{V,\text{even}}[\vec{\varphi}'] \rangle_0.$$

Using the form of $S_{V,\text{even}}[\vec{\varphi}']$ shown in Equation (A.3), and if we already replace the expectation values of squared fields by σ_0 or $\sigma_{\mathfrak{N}}$ whenever possible, as well as replace $\langle \varphi'_1(n'_\mu)\varphi'_1(n''_\mu) \rangle_0$ by $g_0(n'_\mu - n''_\mu)$, we get

$$\begin{aligned} \langle \varphi'_1(n'_\mu)\varphi'_1(n''_\mu)S_V[\vec{\varphi}'] \rangle_0 &= \sum_{n_\mu}^{N^d} \left\{ \frac{\alpha - \alpha_0 + \lambda v_0^2}{2} \langle \varphi_1'^2(n_\mu)\varphi'_1(n'_\mu)\varphi'_1(n''_\mu) \rangle_0 + \right. \\ &\quad + \frac{\alpha - \alpha_0 + \lambda v_0^2}{2} (\mathfrak{N} - 2) \sigma_0^2 g_0(n'_\mu - n''_\mu) + \\ &\quad + \frac{\alpha - \alpha_{\mathfrak{N}} + 3\lambda v_0^2}{2} \sigma_{\mathfrak{N}}^2 g_0(n'_\mu - n''_\mu) + \\ &\quad + \frac{\lambda}{4} \langle \varphi_1'^4(n_\mu)\varphi'_1(n'_\mu)\varphi'_1(n''_\mu) \rangle_0 + \\ &\quad + \frac{\lambda}{2} (\mathfrak{N} - 2) \sigma_0^2 \langle \varphi_1'^2(n_\mu)\varphi'_1(n'_\mu)\varphi'_1(n''_\mu) \rangle_0 + \\ &\quad + \frac{\lambda}{2} \sigma_{\mathfrak{N}}^2 \langle \varphi_1'^2(n_\mu)\varphi'_1(n'_\mu)\varphi'_1(n''_\mu) \rangle_0 + \\ &\quad + \frac{\lambda}{2} (\mathfrak{N} - 2) \sigma_0^2 \sigma_{\mathfrak{N}}^2 g_0(n'_\mu - n''_\mu) + \\ &\quad + \frac{\lambda}{4} \left\langle \left[\sum_{i=2}^{\mathfrak{N}-1} \varphi_i'^2(n_\mu) \right]^2 \right\rangle_0 g_0(n'_\mu - n''_\mu) + \\ &\quad \left. + \frac{\lambda}{4} \langle \varphi_{\mathfrak{N}}'^4(n_\mu) \rangle_0 g_0(n'_\mu - n''_\mu) \right\}. \end{aligned}$$

We may now use the known value of the expectation value of the squared sum. From Appendix B, Equation (B.14), we get

$$\left\langle \left[\sum_{i=2}^{\mathfrak{N}-1} \varphi_i'^2(n_\mu) \right]^2 \right\rangle_0 = \mathfrak{N}(\mathfrak{N} - 2) \sigma_0^4.$$

We may also use the fact that it can be shown that

$$\begin{aligned} \langle \varphi_{\mathfrak{N}}'^4(n_\mu) \rangle_0 &= 3\sigma_{\mathfrak{N}}^4, \\ \langle \varphi_1'^2(n_\mu)\varphi'_1(n'_\mu)\varphi'_1(n''_\mu) \rangle_0 &= \sigma_0^2 g_0(n'_\mu - n''_\mu) + 2 g_0(n_\mu - n'_\mu) g_0(n_\mu - n''_\mu), \\ \langle \varphi_1'^4(n_\mu)\varphi'_1(n'_\mu)\varphi'_1(n''_\mu) \rangle_0 &= 3\sigma_0^4 g_0(n'_\mu - n''_\mu) + 12\sigma_0^2 g_0(n_\mu - n'_\mu) g_0(n_\mu - n''_\mu), \end{aligned}$$

also found in Appendix B, Equations (B.8), (B.10) and (B.12), in order to write for our expectation value

$$\begin{aligned}
\langle \varphi'_1(n'_\mu) \varphi'_1(n''_\mu) S_V[\vec{\varphi}'] \rangle_0 &= \sum_{n_\mu}^{N^d} \left\{ \frac{\alpha - \alpha_0 + \lambda v_0^2}{2} \sigma_0^2 g_0(n'_\mu - n''_\mu) + \right. \\
&\quad + [\alpha - \alpha_0 + \lambda v_0^2] g_0(n_\mu - n'_\mu) g_0(n_\mu - n''_\mu) + \\
&\quad + \frac{\alpha - \alpha_0 + \lambda v_0^2}{2} (\mathfrak{N} - 2) \sigma_0^2 g_0(n'_\mu - n''_\mu) + \\
&\quad + \frac{\alpha - \alpha_{\mathfrak{N}} + 3\lambda v_0^2}{2} \sigma_{\mathfrak{N}}^2 g_0(n'_\mu - n''_\mu) + \\
&\quad + \frac{3\lambda}{4} \sigma_0^4 g_0(n'_\mu - n''_\mu) + \\
&\quad + 3\lambda \sigma_0^2 g_0(n_\mu - n'_\mu) g_0(n_\mu - n''_\mu) + \\
&\quad + \frac{\lambda}{2} (\mathfrak{N} - 2) \sigma_0^4 g_0(n'_\mu - n''_\mu) + \\
&\quad + \lambda (\mathfrak{N} - 2) \sigma_0^2 g_0(n_\mu - n'_\mu) g_0(n_\mu - n''_\mu) + \\
&\quad + \frac{\lambda}{2} \sigma_{\mathfrak{N}}^2 \sigma_0^2 g_0(n'_\mu - n''_\mu) + \\
&\quad + \lambda \sigma_{\mathfrak{N}}^2 g_0(n_\mu - n'_\mu) g_0(n_\mu - n''_\mu) + \\
&\quad + \frac{\lambda}{2} (\mathfrak{N} - 2) \sigma_0^2 \sigma_{\mathfrak{N}}^2 g_0(n'_\mu - n''_\mu) + \\
&\quad + \frac{\lambda}{4} \mathfrak{N} (\mathfrak{N} - 2) \sigma_0^4 g_0(n'_\mu - n''_\mu) + \\
&\quad \left. + \frac{3\lambda}{4} \sigma_{\mathfrak{N}}^4 g_0(n'_\mu - n''_\mu) \right\}.
\end{aligned}$$

Next we group all terms containing $g_0(n'_\mu - n''_\mu)$ and simplify to get

$$\begin{aligned}
&\langle \varphi'_1(n'_\mu) \varphi'_1(n''_\mu) S_V[\vec{\varphi}'] \rangle_0 \\
&= \sum_{n_\mu}^{N^d} \left[\frac{\alpha - \alpha_0 + \lambda v_0^2}{2} (\mathfrak{N} - 1) \sigma_0^2 + \frac{\alpha - \alpha_{\mathfrak{N}} + 3\lambda v_0^2}{2} \sigma_{\mathfrak{N}}^2 + \right. \\
&\quad \left. + \frac{\lambda}{4} (\mathfrak{N}^2 - 1) \sigma_0^4 + \frac{\lambda}{2} (\mathfrak{N} - 1) \sigma_0^2 \sigma_{\mathfrak{N}}^2 + \frac{3\lambda}{4} \sigma_{\mathfrak{N}}^4 \right] g_0(n'_\mu - n''_\mu) \\
&\quad + \sum_{n_\mu}^{N^d} \left\{ [\alpha - \alpha_0 + \lambda v_0^2] + \lambda (\mathfrak{N} + 1) \sigma_0^2 + \lambda \sigma_{\mathfrak{N}}^2 \right\} g_0(n_\mu - n'_\mu) g_0(n_\mu - n''_\mu).
\end{aligned}$$

The sum over n_μ can now be done in all terms in the first group, yielding

$$\begin{aligned}
& \langle \varphi'_1(n'_\mu) \varphi'_1(n''_\mu) S_V[\vec{\varphi}'] \rangle_0 \\
&= N^d \left[\frac{\alpha - \alpha_0 + \lambda v_0^2}{2} (\mathfrak{N} - 1) \sigma_0^2 + \frac{\alpha - \alpha_{\mathfrak{N}} + 3\lambda v_0^2}{2} \sigma_{\mathfrak{N}}^2 + \right. \\
&\quad \left. + \frac{\lambda}{4} (\mathfrak{N}^2 - 1) \sigma_0^4 + \frac{\lambda}{2} (\mathfrak{N} - 1) \sigma_0^2 \sigma_{\mathfrak{N}}^2 + \frac{3\lambda}{4} \sigma_{\mathfrak{N}}^4 \right] g_0(n'_\mu - n''_\mu) \\
&\quad + [\alpha - \alpha_0 + \lambda v_0^2 + \lambda(\mathfrak{N} + 1) \sigma_0^2 + \lambda \sigma_{\mathfrak{N}}^2] \sum_{n_\mu}^{N^d} g_0(n_\mu - n'_\mu) g_0(n_\mu - n''_\mu).
\end{aligned}$$

We must now perform the sum indicated. This is easily done using Fourier transforms. From Appendix B, Equation (B.4), we get

$$\sum_{n_\mu}^{N^d} g_0(n_\mu - n'_\mu) g_0(n_\mu - n''_\mu) = \frac{1}{N^d} \sum_{k_\mu}^{N^d} \frac{e^{-\mathbf{i}(2\pi/N) \sum_\mu^d k_\mu (n'_\mu - n''_\mu)}}{[\rho^2(k_\mu) + \alpha_0]^2},$$

which is expressed as a Fourier transform, with the general structure of a two-point function. We have therefore the final result,

$$\begin{aligned}
& \langle \varphi'_1(n'_\mu) \varphi'_1(n''_\mu) S_V[\vec{\varphi}'] \rangle_0 \\
&= N^d \left[\frac{\alpha - \alpha_0 + \lambda v_0^2}{2} (\mathfrak{N} - 1) \sigma_0^2 + \frac{\alpha - \alpha_{\mathfrak{N}} + 3\lambda v_0^2}{2} \sigma_{\mathfrak{N}}^2 + \right. \\
&\quad \left. + \frac{\lambda}{4} (\mathfrak{N}^2 - 1) \sigma_0^4 + \frac{\lambda}{2} (\mathfrak{N} - 1) \sigma_0^2 \sigma_{\mathfrak{N}}^2 + \frac{3\lambda}{4} \sigma_{\mathfrak{N}}^4 \right] g_0(n'_\mu - n''_\mu) \\
&\quad + [\alpha - \alpha_0 + \lambda v_0^2 + \lambda(\mathfrak{N} + 1) \sigma_0^2 + \lambda \sigma_{\mathfrak{N}}^2] \frac{1}{N^d} \sum_{k_\mu}^{N^d} \frac{e^{-\mathbf{i}(2\pi/N) \sum_\mu^d k_\mu (n'_\mu - n''_\mu)}}{[\rho^2(k_\mu) + \alpha_0]^2}.
\end{aligned} \tag{A.6}$$

A.4 The Expectation Value of $\varphi'_{\mathfrak{N}}(n'_\mu) \varphi'_{\mathfrak{N}}(n''_\mu) S_V[\vec{\varphi}']$

We now calculate the expectation value

$$\langle \varphi'_{\mathfrak{N}}(n'_\mu) \varphi'_{\mathfrak{N}}(n''_\mu) S_V[\vec{\varphi}'] \rangle_0.$$

Once again only the field-even part of the action will yield a non-zero result, so that we have

$$\langle \varphi'_{\mathfrak{N}}(n'_\mu) \varphi'_{\mathfrak{N}}(n''_\mu) S_V[\vec{\varphi}'] \rangle_0 = \langle \varphi'_{\mathfrak{N}}(n'_\mu) \varphi'_{\mathfrak{N}}(n''_\mu) S_{V,\text{even}}[\vec{\varphi}'] \rangle_0.$$

Using the form of $S_{V,\text{even}}[\vec{\varphi}']$ shown in Equation (A.2), and if we already replace the expectation values of squared fields by σ_0 or $\sigma_{\mathfrak{N}}$ whenever possible, as well as replace $\langle \varphi'_{\mathfrak{N}}(n'_\mu) \varphi'_{\mathfrak{N}}(n''_\mu) \rangle_0$ by $g_{\mathfrak{N}}(n'_\mu - n''_\mu)$, we get

$$\begin{aligned}
\langle \varphi'_{\mathfrak{N}}(n'_\mu) \varphi'_{\mathfrak{N}}(n''_\mu) S_V[\vec{\varphi}'] \rangle_0 &= \sum_{n_\mu}^{N^d} \left\{ \frac{\alpha - \alpha_0 + \lambda v_0^2}{2} (\mathfrak{N} - 1) \sigma_0^2 g_{\mathfrak{N}}(n'_\mu - n''_\mu) + \right. \\
&\quad \left. + \frac{\alpha - \alpha_{\mathfrak{N}} + 3\lambda v_0^2}{2} \langle \varphi'^2_{\mathfrak{N}}(n_\mu) \varphi'_{\mathfrak{N}}(n'_\mu) \varphi'_{\mathfrak{N}}(n''_\mu) \rangle_0 + \right. \\
&\quad \left. + \frac{\lambda}{4} \left\langle \left[\sum_{i=1}^{\mathfrak{N}-1} \varphi'^2_i(n_\mu) \right]^2 \right\rangle_0 g_{\mathfrak{N}}(n'_\mu - n''_\mu) + \right.
\end{aligned}$$

$$\begin{aligned}
& + \frac{\lambda}{2} (\mathfrak{N} - 1) \sigma_0^2 \langle \varphi_{\mathfrak{N}}'^2(n_\mu) \varphi_{\mathfrak{N}}'(n'_\mu) \varphi_{\mathfrak{N}}'(n''_\mu) \rangle_0 + \\
& + \frac{\lambda}{4} \langle \varphi_{\mathfrak{N}}'^4(n_\mu) \varphi_{\mathfrak{N}}'(n'_\mu) \varphi_{\mathfrak{N}}'(n''_\mu) \rangle_0 \Bigg\}.
\end{aligned}$$

We may now use the known value of the expectation value of the squared sum, found in Appendix B, Equation (B.15),

$$\left\langle \left[\sum_{i=1}^{\mathfrak{N}-1} \varphi_i'^2(n_\mu) \right]^2 \right\rangle_0 = (\mathfrak{N} + 1)(\mathfrak{N} - 1) \sigma_0^4,$$

as well as the fact that it can be shown that

$$\begin{aligned}
\langle \varphi_{\mathfrak{N}}'^2(n_\mu) \varphi_{\mathfrak{N}}'(n'_\mu) \varphi_{\mathfrak{N}}'(n''_\mu) \rangle_0 &= \sigma_{\mathfrak{N}}^2 g_{\mathfrak{N}}(n'_\mu - n''_\mu) + 2 g_{\mathfrak{N}}(n_\mu - n'_\mu) g_{\mathfrak{N}}(n_\mu - n''_\mu), \\
\langle \varphi_{\mathfrak{N}}'^4(n_\mu) \varphi_{\mathfrak{N}}'(n'_\mu) \varphi_{\mathfrak{N}}'(n''_\mu) \rangle_0 &= 3 \sigma_{\mathfrak{N}}^4 g_{\mathfrak{N}}(n'_\mu - n''_\mu) + 12 \sigma_{\mathfrak{N}}^2 g_{\mathfrak{N}}(n_\mu - n'_\mu) g_{\mathfrak{N}}(n_\mu - n''_\mu),
\end{aligned}$$

as one can also see in Appendix B, Equations (B.11) and (B.13), in order to write for our expectation value

$$\begin{aligned}
\langle \varphi_{\mathfrak{N}}'(n'_\mu) \varphi_{\mathfrak{N}}'(n''_\mu) S_V[\vec{\varphi}'] \rangle_0 &= \sum_{n_\mu}^{N^d} \Bigg\{ \frac{\alpha - \alpha_0 + \lambda v_0^2}{2} (\mathfrak{N} - 1) \sigma_0^2 g_{\mathfrak{N}}(n'_\mu - n''_\mu) + \\
& + \frac{\alpha - \alpha_{\mathfrak{N}} + 3 \lambda v_0^2}{2} \sigma_{\mathfrak{N}}^2 g_{\mathfrak{N}}(n'_\mu - n''_\mu) + \\
& + [\alpha - \alpha_{\mathfrak{N}} + 3 \lambda v_0^2] g_{\mathfrak{N}}(n_\mu - n'_\mu) g_{\mathfrak{N}}(n_\mu - n''_\mu) + \\
& + \frac{\lambda}{4} (\mathfrak{N}^2 - 1) \sigma_0^4 g_{\mathfrak{N}}(n'_\mu - n''_\mu) + \\
& + \frac{\lambda}{2} (\mathfrak{N} - 1) \sigma_0^2 \sigma_{\mathfrak{N}}^2 g_{\mathfrak{N}}(n'_\mu - n''_\mu) + \\
& + \lambda (\mathfrak{N} - 1) \sigma_0^2 g_{\mathfrak{N}}(n_\mu - n'_\mu) g_{\mathfrak{N}}(n_\mu - n''_\mu) + \\
& + \frac{3 \lambda}{4} \sigma_{\mathfrak{N}}^4 g_{\mathfrak{N}}(n'_\mu - n''_\mu) + \\
& + 3 \lambda \sigma_{\mathfrak{N}}^2 g_{\mathfrak{N}}(n_\mu - n'_\mu) g_{\mathfrak{N}}(n_\mu - n''_\mu) \Bigg\}.
\end{aligned}$$

Next we group all terms containing $g_{\mathfrak{N}}(n'_\mu - n''_\mu)$ and simplify to get

$$\begin{aligned}
& \langle \varphi_{\mathfrak{N}}'(n'_\mu) \varphi_{\mathfrak{N}}'(n''_\mu) S_V[\vec{\varphi}'] \rangle_0 \\
&= \sum_{n_\mu}^{N^d} \left[\frac{\alpha - \alpha_0 + \lambda v_0^2}{2} (\mathfrak{N} - 1) \sigma_0^2 + \frac{\alpha - \alpha_{\mathfrak{N}} + 3 \lambda v_0^2}{2} \sigma_{\mathfrak{N}}^2 + \right. \\
& \quad \left. + \frac{\lambda}{4} (\mathfrak{N}^2 - 1) \sigma_0^4 + \frac{\lambda}{2} (\mathfrak{N} - 1) \sigma_0^2 \sigma_{\mathfrak{N}}^2 + \frac{3 \lambda}{4} \sigma_{\mathfrak{N}}^4 \right] g_{\mathfrak{N}}(n'_\mu - n''_\mu)
\end{aligned}$$

$$+ \sum_{n_\mu}^{N^d} \left\{ [\alpha - \alpha_{\mathfrak{N}} + 3\lambda v_0^2] + \lambda(\mathfrak{N} - 1)\sigma_0^2 + 3\lambda\sigma_{\mathfrak{N}}^2 \right\} g_{\mathfrak{N}}(n_\mu - n'_\mu) g_{\mathfrak{N}}(n_\mu - n''_\mu).$$

The sum over n_μ can now be done in all terms of the first group, yielding

$$\begin{aligned} & \langle \varphi'_{\mathfrak{N}}(n'_\mu) \varphi'_{\mathfrak{N}}(n''_\mu) S_V[\vec{\varphi}] \rangle_0 \\ &= N^d \left[\frac{\alpha - \alpha_0 + \lambda v_0^2}{2} (\mathfrak{N} - 1) \sigma_0^2 + \frac{\alpha - \alpha_{\mathfrak{N}} + 3\lambda v_0^2}{2} \sigma_{\mathfrak{N}}^2 + \right. \\ & \quad \left. + \frac{\lambda}{4} (\mathfrak{N}^2 - 1) \sigma_0^4 + \frac{\lambda}{2} (\mathfrak{N} - 1) \sigma_0^2 \sigma_{\mathfrak{N}}^2 + \frac{3\lambda}{4} \sigma_{\mathfrak{N}}^4 \right] g_{\mathfrak{N}}(n'_\mu - n''_\mu) \\ & \quad + [\alpha - \alpha_{\mathfrak{N}} + 3\lambda v_0^2 + \lambda(\mathfrak{N} - 1)\sigma_0^2 + 3\lambda\sigma_{\mathfrak{N}}^2] \sum_{n_\mu}^{N^d} g_{\mathfrak{N}}(n_\mu - n'_\mu) g_{\mathfrak{N}}(n_\mu - n''_\mu). \end{aligned}$$

We must now perform the sum indicated. We get from Appendix B, Equation (B.5),

$$\sum_{n_\mu}^{N^d} g_{\mathfrak{N}}(n_\mu - n'_\mu) g_{\mathfrak{N}}(n_\mu - n''_\mu) = \frac{1}{N^d} \sum_{k_\mu}^{N^d} \frac{e^{-\mathfrak{I}(2\pi/N) \sum_\mu^d k_\mu (n'_\mu - n''_\mu)}}{[\rho^2(k_\mu) + \alpha_{\mathfrak{N}}]^2}.$$

We have therefore the final result

$$\begin{aligned} & \langle \varphi'_{\mathfrak{N}}(n'_\mu) \varphi'_{\mathfrak{N}}(n''_\mu) S_V[\vec{\varphi}] \rangle_0 \\ &= N^d \left[\frac{\alpha - \alpha_0 + \lambda v_0^2}{2} (\mathfrak{N} - 1) \sigma_0^2 + \frac{\alpha - \alpha_{\mathfrak{N}} + 3\lambda v_0^2}{2} \sigma_{\mathfrak{N}}^2 + \right. \\ & \quad \left. + \frac{\lambda}{4} (\mathfrak{N}^2 - 1) \sigma_0^4 + \frac{\lambda}{2} (\mathfrak{N} - 1) \sigma_0^2 \sigma_{\mathfrak{N}}^2 + \frac{3\lambda}{4} \sigma_{\mathfrak{N}}^4 \right] g_{\mathfrak{N}}(n'_\mu - n''_\mu) \\ & \quad + [\alpha - \alpha_{\mathfrak{N}} + 3\lambda v_0^2 + \lambda(\mathfrak{N} - 1)\sigma_0^2 + 3\lambda\sigma_{\mathfrak{N}}^2] \frac{1}{N^d} \sum_{k_\mu}^{N^d} \frac{e^{-\mathfrak{I}(2\pi/N) \sum_\mu^d k_\mu (n'_\mu - n''_\mu)}}{[\rho^2(k_\mu) + \alpha_{\mathfrak{N}}]^2}. \end{aligned} \tag{A.7}$$

B Table of Integrals and Lattice Sums

We give here a series of formulas and derivations involving Gaussian integrals, Gaussian expectation values and lattice sums, in the context the model discussed in this paper, which are used for the calculations presented. All these can be derived from the basic result in momentum space

$$\langle \tilde{\varphi}'_i(k_\mu) \tilde{\varphi}'_i^*(k_\mu) \rangle_0 = \frac{1}{N^d} \frac{1}{\rho^2(k_\mu) + \alpha_i}, \tag{B.1}$$

where α_i is either α_0 or $\alpha_{\mathfrak{N}}$, depending on the field component involved, and where $\rho^2(k_\mu)$ are the eigenvalues of the discrete Laplacian on the lattice, which are given by

$$\rho^2(k_\mu) = 4 \left[\sin^2 \left(\frac{\pi k_1}{N} \right) + \dots + \sin^2 \left(\frac{\pi k_d}{N} \right) \right].$$

Since in the measure of $S_0[\vec{\varphi}]$ the modes are decoupled in momentum space, the same expectation value with two different momenta k_μ and k'_μ is zero by simple parity arguments. We use the notation for the two-point functions in position space,

$$\begin{aligned}
g_0(n_\mu - n'_\mu) &= \langle \varphi'_i(n_\mu) \varphi'_i(n'_\mu) \rangle_0, \\
g_{\mathfrak{N}}(n_\mu - n'_\mu) &= \langle \varphi'_{\mathfrak{N}}(n_\mu) \varphi'_{\mathfrak{N}}(n'_\mu) \rangle_0,
\end{aligned}$$

for $i = 1, \dots, \mathfrak{N}-1$. These are, of course, the inverse Fourier transforms of the corresponding two-point functions in momentum space,

$$\begin{aligned}
g_0(n_\mu - n'_\mu) &= \sum_{k_\mu}^{N^d} e^{-\mathbf{i}(2\pi/N) \sum_\mu k_\mu (n_\mu - n'_\mu)} \langle \tilde{\varphi}'_i(k_\mu) \tilde{\varphi}'_{i*}(k_\mu) \rangle_0, \\
g_{\mathfrak{N}}(n_\mu - n'_\mu) &= \sum_{k_\mu}^{N^d} e^{-\mathbf{i}(2\pi/N) \sum_\mu k_\mu (n_\mu - n'_\mu)} \langle \tilde{\varphi}'_{\mathfrak{N}}(k_\mu) \tilde{\varphi}'_{\mathfrak{N}*}(k_\mu) \rangle_0.
\end{aligned}$$

In order to write this explicitly we may use the Fourier transforms of the fields, for example in the case of the $\varphi'_{\mathfrak{N}}(n_\mu)$ field component,

$$\begin{aligned}
g_{\mathfrak{N}}(n_\mu - n'_\mu) &= \langle \varphi'_{\mathfrak{N}}(n_\mu) \varphi'_{\mathfrak{N}}(n'_\mu) \rangle_0 \\
&= \sum_{k_\mu}^{N^d} \sum_{k'_\mu}^{N^d} e^{-\mathbf{i}(2\pi/N) \sum_\mu (k_\mu n_\mu + k'_\mu n'_\mu)} \langle \tilde{\varphi}'_{\mathfrak{N}}(k_\mu) \tilde{\varphi}'_{\mathfrak{N}}(k'_\mu) \rangle_0.
\end{aligned}$$

The expectation value in momentum space is non-zero only if we have $k'_\mu = -k_\mu$, in which case we have the result, which can be obtained from Equation (B.1) above,

$$\begin{aligned}
\langle \tilde{\varphi}'_{\mathfrak{N}}(k_\mu) \tilde{\varphi}'_{\mathfrak{N}}(-k_\mu) \rangle_0 &= \langle \tilde{\varphi}'_{\mathfrak{N}}(k_\mu) \tilde{\varphi}'_{\mathfrak{N}*}(k_\mu) \rangle_0 \\
&= \frac{1}{N^d} \frac{1}{\rho^2(k_\mu) + \alpha_{\mathfrak{N}}}.
\end{aligned}$$

This eliminates one of the momentum-space sums, and thus we get

$$g_{\mathfrak{N}}(n_\mu - n'_\mu) = \frac{1}{N^d} \sum_{k_\mu}^{N^d} \frac{e^{-\mathbf{i}(2\pi/N) \sum_\mu k_\mu (n_\mu - n'_\mu)}}{\rho^2(k_\mu) + \alpha_{\mathfrak{N}}}, \quad (\text{B.2})$$

which is just the statement that $g_{\mathfrak{N}}(n_\mu - n'_\mu)$ is the inverse Fourier transform of the momentum-space propagator. Note that this is necessarily real, and that therefore the imaginary part of the right-hand side vanishes. In a completely similar way, we have the corresponding result for the other field $\mathfrak{N}-1$ components, with $i \neq \mathfrak{N}$,

$$g_0(n_\mu - n'_\mu) = \frac{1}{N^d} \sum_{k_\mu}^{N^d} \frac{e^{-\mathbf{i}(2\pi/N) \sum_\mu k_\mu (n_\mu - n'_\mu)}}{\rho^2(k_\mu) + \alpha_0}.$$

The following sum involving $g_0(n_\mu - n'_\mu)$ can also be easily calculated, using once more the Fourier transforms,

$$\sum_{n_\mu}^{N^d} g_{\mathfrak{N}}(n_\mu - n'_\mu) = \frac{1}{N^d} \sum_{n_\mu}^{N^d} \sum_{k_\mu}^{N^d} \frac{e^{-\mathbf{i}(2\pi/N) \sum_\mu k_\mu (n'_\mu - n''_\mu)}}{\rho^2(k_\mu) + \alpha_{\mathfrak{N}}}.$$

The orthogonality relation can be used to simplify this expression, and thus we get

$$\begin{aligned}
\sum_{n_\mu}^{N^d} g_{\mathfrak{N}}(n_\mu - n'_\mu) &= \sum_{k_\mu}^{N^d} \delta^d(k_\mu, 0_\mu) \frac{e^{\mathfrak{I}(2\pi/N) \sum_\mu^d k_\mu n'_\mu}}{\rho^2(k_\mu) + \alpha_{\mathfrak{N}}} \\
&= \frac{e^0}{\rho^2(0) + \alpha_{\mathfrak{N}}} \\
&= \frac{1}{\alpha_{\mathfrak{N}}}.
\end{aligned}$$

This is simply the zero-mode of the propagator. The same can be done for the other components of the field, so we conclude that

$$\begin{aligned}
\sum_{n_\mu}^{N^d} g_0(n_\mu - n'_\mu) &= \frac{1}{\alpha_0}, \\
\sum_{n_\mu}^{N^d} g_{\mathfrak{N}}(n_\mu - n'_\mu) &= \frac{1}{\alpha_{\mathfrak{N}}}.
\end{aligned} \tag{B.3}$$

A similar sum with two chained factors of $g_{\mathfrak{N}}(n_\mu - n'_\mu)$ can be calculated in a similar way. Using the Fourier expressions of $g_0(n_\mu - n'_\mu)$ and $g_0(n_\mu - n''_\mu)$ we get

$$\begin{aligned}
&\sum_{n_\mu}^{N^d} g_0(n_\mu - n'_\mu) g_0(n_\mu - n''_\mu) \\
&= \sum_{n_\mu}^{N^d} \frac{1}{N^{2d}} \sum_{k'_\mu}^{N^d} \sum_{k''_\mu}^{N^d} \frac{e^{-\mathfrak{I}(2\pi/N) \sum_\mu^d [k'_\mu(n_\mu - n'_\mu) + k''_\mu(n_\mu - n''_\mu)]}}{[\rho^2(k'_\mu) + \alpha_0][\rho^2(k''_\mu) + \alpha_0]} \\
&= \frac{1}{N^{2d}} \sum_{k'_\mu}^{N^d} \sum_{k''_\mu}^{N^d} \frac{e^{\mathfrak{I}(2\pi/N) \sum_\mu^d (k'_\mu n'_\mu + k''_\mu n''_\mu)}}{[\rho^2(k'_\mu) + \alpha_0][\rho^2(k''_\mu) + \alpha_0]} \sum_{n_\mu}^{N^d} e^{-\mathfrak{I}(2\pi/N) \sum_\mu^d (k'_\mu + k''_\mu) n_\mu} \\
&= \frac{1}{N^d} \sum_{k'_\mu}^{N^d} \sum_{k''_\mu}^{N^d} \delta^d(k'_\mu, -k''_\mu) \frac{e^{\mathfrak{I}(2\pi/N) \sum_\mu^d (k'_\mu n'_\mu + k''_\mu n''_\mu)}}{[\rho^2(k'_\mu) + \alpha_0][\rho^2(k''_\mu) + \alpha_0]} \\
&= \frac{1}{N^d} \sum_{k''_\mu}^{N^d} \frac{e^{-\mathfrak{I}(2\pi/N) \sum_\mu^d (k''_\mu n'_\mu - k''_\mu n''_\mu)}}{[\rho^2(k''_\mu) + \alpha_0]^2}.
\end{aligned}$$

We see therefore that we get the sum expressed as a Fourier transform, with the general structure of a two-point function,

$$\sum_{n_\mu}^{N^d} g_0(n_\mu - n'_\mu) g_0(n_\mu - n''_\mu) = \frac{1}{N^d} \sum_{k_\mu}^{N^d} \frac{e^{-\mathfrak{I}(2\pi/N) \sum_\mu^d k_\mu (n'_\mu - n''_\mu)}}{[\rho^2(k_\mu) + \alpha_0]^2}. \tag{B.4}$$

A similar result is true, of course, for the remaining field component

$$\sum_{n_\mu}^{N^d} g_{\mathfrak{N}}(n_\mu - n'_\mu) g_{\mathfrak{N}}(n_\mu - n''_\mu) = \frac{1}{N^d} \sum_{k_\mu}^{N^d} \frac{e^{-\mathfrak{I}(2\pi/N) \sum_\mu^d k_\mu (n'_\mu - n''_\mu)}}{[\rho^2(k_\mu) + \alpha_{\mathfrak{N}}]^2}. \tag{B.5}$$

The squared dispersions, also referred to as widths or variances of the fields at a given site, are denoted as

$$\begin{aligned}\sigma_0^2 &= \langle \varphi_i'^2(n_\mu) \rangle_0, \\ \sigma_{\mathfrak{N}}^2 &= \langle \varphi_{\mathfrak{N}}'^2(n_\mu) \rangle_0,\end{aligned}$$

for $i = 1, \dots, \mathfrak{N} - 1$. Using the expression of the two-point function in terms of Fourier components we may write these explicitly as

$$\sigma_0^2 = \frac{1}{N^d} \sum_{k_\mu} \frac{1}{\rho^2(k_\mu) + \alpha_0}, \quad (\text{B.6})$$

$$\sigma_{\mathfrak{N}}^2 = \frac{1}{N^d} \sum_{k_\mu} \frac{1}{\rho^2(k_\mu) + \alpha_{\mathfrak{N}}}. \quad (\text{B.7})$$

In terms of these quantities the following decompositions of higher-point functions can be established, always for $i = 1, \dots, \mathfrak{N} - 1$,

$$\begin{aligned}\langle \varphi_i'^4(n_\mu) \rangle_0 &= 3\sigma_0^4, \\ \langle \varphi_{\mathfrak{N}}'^4(n_\mu) \rangle_0 &= 3\sigma_{\mathfrak{N}}^4,\end{aligned} \quad (\text{B.8})$$

$$\begin{aligned}\langle \varphi_i'^3(n_\mu) \varphi_i'(n'_\mu) \rangle_0 &= 3\sigma_0^2 g_0(n_\mu - n'_\mu), \\ \langle \varphi_{\mathfrak{N}}'^3(n_\mu) \varphi_{\mathfrak{N}}'(n'_\mu) \rangle_0 &= 3\sigma_{\mathfrak{N}}^2 g_{\mathfrak{N}}(n_\mu - n'_\mu),\end{aligned} \quad (\text{B.9})$$

$$\begin{aligned}\langle \varphi_i'^2(n_\mu) \varphi_i'(n'_\mu) \varphi_i'(n''_\mu) \rangle_0 &= \sigma_0^2 g_0(n'_\mu - n''_\mu) + \\ &\quad + 2 g_0(n_\mu - n'_\mu) g_0(n_\mu - n''_\mu),\end{aligned} \quad (\text{B.10})$$

$$\begin{aligned}\langle \varphi_{\mathfrak{N}}'^2(n_\mu) \varphi_{\mathfrak{N}}'(n'_\mu) \varphi_{\mathfrak{N}}'(n''_\mu) \rangle_0 &= \sigma_{\mathfrak{N}}^2 g_{\mathfrak{N}}(n'_\mu - n''_\mu) + \\ &\quad + 2 g_{\mathfrak{N}}(n_\mu - n'_\mu) g_{\mathfrak{N}}(n_\mu - n''_\mu),\end{aligned} \quad (\text{B.11})$$

$$\begin{aligned}\langle \varphi_i'^4(n_\mu) \varphi_i'(n'_\mu) \varphi_i'(n''_\mu) \rangle_0 &= 3\sigma_0^4 g_0(n'_\mu - n''_\mu) + \\ &\quad + 12\sigma_0^2 g_0(n_\mu - n'_\mu) g_0(n_\mu - n''_\mu),\end{aligned} \quad (\text{B.12})$$

$$\begin{aligned}\langle \varphi_{\mathfrak{N}}'^4(n_\mu) \varphi_{\mathfrak{N}}'(n'_\mu) \varphi_{\mathfrak{N}}'(n''_\mu) \rangle_0 &= 3\sigma_{\mathfrak{N}}^4 g_{\mathfrak{N}}(n'_\mu - n''_\mu) + \\ &\quad + 12\sigma_{\mathfrak{N}}^2 g_{\mathfrak{N}}(n_\mu - n'_\mu) g_{\mathfrak{N}}(n_\mu - n''_\mu).\end{aligned} \quad (\text{B.13})$$

It is also not difficult to expand and calculate the following sums,

$$\begin{aligned}\left\langle \left[\sum_{i=2}^{\mathfrak{N}-1} \varphi_i'^2(n_\mu) \right]^2 \right\rangle_0 &= (\mathfrak{N}-2) \langle \varphi_1'^4(n_\mu) \rangle_0 + (\mathfrak{N}-2)(\mathfrak{N}-3) \langle \varphi_1'^2(n_\mu) \rangle_0^2 \\ &= 3(\mathfrak{N}-2)\sigma_0^4 + (\mathfrak{N}-2)(\mathfrak{N}-3)\sigma_0^4 \\ &= (\mathfrak{N})(\mathfrak{N}-2)\sigma_0^4, \\ \left\langle \left[\sum_{i=1}^{\mathfrak{N}-1} \varphi_i'^2(n_\mu) \right]^2 \right\rangle_0 &= (\mathfrak{N}-1) \langle \varphi_1'^4(n_\mu) \rangle_0 + (\mathfrak{N}-1)(\mathfrak{N}-2) \langle \varphi_1'^2(n_\mu) \rangle_0^2 \\ &= 3(\mathfrak{N}-1)\sigma_0^4 + (\mathfrak{N}-1)(\mathfrak{N}-2)\sigma_0^4 \\ &= (\mathfrak{N}+1)(\mathfrak{N}-1)\sigma_0^4,\end{aligned}$$

so that we get the results

$$\left\langle \left[\sum_{i=2}^{\mathfrak{N}-1} \varphi_i'^2(n_\mu) \right]^2 \right\rangle_0 = \mathfrak{N}(\mathfrak{N}-2)\sigma_0^4, \quad (\text{B.14})$$

$$\left\langle \left[\sum_{i=1}^{\mathfrak{N}-1} \varphi_i'^2(n_\mu) \right]^2 \right\rangle_0 = (\mathfrak{N}+1)(\mathfrak{N}-1)\sigma_0^4. \quad (\text{B.15})$$